Vector Space of Linear Transformations

Problems:

1) Let $S, T : R^2 \to R^2$ by $S(x,y) = (x+3y, 2x)$ and $T(x,y) = (y, x+2y)$ and $\alpha = \{(1,1), (1,2)\}$ is any basis then find $[S+T]_\alpha$, $[S \circ T]_\alpha$ and $[2T-3S]_\alpha$.

Solution:

Given, $S(x,y) = (x+3y, 2x)$ and $\alpha = \{\underset{v_1}{(1,1)}, \underset{v_2}{(1,2)}\}$

$S(v_1) = S(1,1) = (4,2)$

$$(4,2) = a_1 v_1 + a_2 v_2$$
$$= a_1(1,1) + a_2(1,2) = (a_1, a_1) + (a_2, 2a_2)$$
$$= (a_1 + a_2, a_1 + 2a_2)$$

$$\Rightarrow \left.\begin{array}{l} a_1 + a_2 = 4 \\ a_1 + 2a_2 = 2 \end{array}\right\} \Rightarrow \begin{array}{l} a_1 = 6 \\ a_2 = -2. \end{array}$$

Similarly, $S(v_2) = S(1,2) = (7,2)$

$$(7,2) = b_1 v_1 + b_2 v_2 = b_1(1,1) + b_2(1,2)$$
$$= (b_1, b_1) + (b_2, 2b_2) = (b_1 + b_2, b_1 + 2b_2)$$

$$\Rightarrow \left.\begin{array}{l} b_1 + b_2 = 7 \\ b_1 + 2b_2 = 2 \end{array}\right\} \Rightarrow b_1 = 12 \text{ and } b_2 = -5.$$

$$\therefore \quad S(v_1) = 6v_1 - 2v_2$$
$$S(v_2) = 12v_1 - 5v_2$$
$$[S]_\alpha = \begin{bmatrix} 6 & 12 \\ -2 & -5 \end{bmatrix}.$$

Similarly, $T(v_1) = T(1,1) = (1,3) = c_1v_1 + c_2v_2$

$$(1,3) = c_1(1,1) + c_2(1,2) = (c_1, c_1) + (c_2, 2c_2)$$
$$= (c_1 + c_2,\ c_1 + 2c_2)$$

$$\Rightarrow \left.\begin{aligned} c_1 + c_2 &= 1 \\ c_1 + 2c_2 &= 3 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= -1 \\ c_2 &= 2. \end{aligned}$$

$$T(v_2) = T(1,2) = (2,5) = d_1v_1 + d_2v_2 = d_1(1,1) + d_2(1,2)$$
$$(2,5) = (d_1, d_1) + (d_2, 2d_2) = (d_1 + d_2,\ d_1 + 2d_2)$$

$$\Rightarrow \left.\begin{aligned} d_1 + d_2 &= 2 \\ d_1 + 2d_2 &= 5 \end{aligned}\right\} \Rightarrow \begin{aligned} d_1 &= -1 \\ d_2 &= 3 \end{aligned}$$

$$\therefore \quad T(v_1) = -v_1 + 2v_2$$
$$T(v_2) = -v_1 + 3v_2$$
$$\therefore \quad [T]_\alpha = \begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix}$$

$$[S+T]_\alpha = [S]_\alpha + [T]_\alpha = \begin{bmatrix} 6 & 12 \\ -2 & -5 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 11 \\ 0 & -2 \end{bmatrix}.$$

$$[S \circ T]_\alpha = [S]_\alpha \circ [T]_\alpha = \begin{bmatrix} 6 & 12 \\ -2 & -5 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 30 \\ -8 & -13 \end{bmatrix}.$$

$$[2T - 3S]_\alpha = [2T]_\alpha - [3S]_\alpha = 2[T]_\alpha - 3[S]_\alpha$$
$$= 2\begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix} - 3\begin{bmatrix} 6 & 12 \\ -2 & -5 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 4 & 6 \end{bmatrix} - \begin{bmatrix} 18 & 36 \\ -6 & -15 \end{bmatrix}$$
$$= \begin{bmatrix} -20 & -38 \\ 10 & 21 \end{bmatrix}.$$

2) Let α be the standard basis for R^3, and let $S, T : R^3 \to R^3$ be two linear transformations given by $S(e_1) = (2,2,1)$, $S(e_2) = (0,1,2)$, $S(e_3) = (-1,2,1)$ and $T(e_1) = (1,0,1)$, $T(e_2) = (0,1,1)$, $T(e_3) = (1,1,2)$.

Compute, $[S+T]_\alpha$, $[2T-S]_\alpha$ and $[T \circ S]_\alpha$.

Solution:

Given, α be the standard basis for R^3.

$$\alpha = \{\underset{v_1}{(1,0,0)}, \underset{v_2}{(0,1,0)}, \underset{v_3}{(0,0,1)}\}.$$

$$S(e_1) = (2,2,1) = a_1 v_1 + a_2 v_2 + a_3 v_3$$
$$= a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1)$$
$$= (a_1, a_2, a_3)$$

$\Rightarrow a_1 = 2, \quad a_2 = 2, \quad a_3 = 1.$

$$S(e_2) = (0,1,2) = b_1 v_1 + b_2 v_2 + b_3 v_3$$
$$= b_1(1,0,0) + b_2(0,1,0) + b_3(0,0,1)$$
$$= (b_1, b_2, b_3)$$

$\Rightarrow b_1 = 0, b_2 = 1, b_3 = 2.$

$$S(e_3) = (-1,2,1) = c_1 v_1 + c_2 v_2 + c_3 v_3$$
$$= c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1)$$
$$= (c_1, c_2, c_3)$$

$\Rightarrow c_1 = -1, \; c_2 = 2, \; c_3 = 1.$

$$\therefore \left.\begin{array}{l} S(e_1) = 2v_1 + 2v_2 + v_3, \\ S(e_2) = 0 \cdot v_1 + v_2 + 2v_3, \\ S(e_3) = -v_1 + 2v_2 + v_3, \end{array}\right\} \Rightarrow [S]_\alpha = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

Similarly, $T(e_1) = (1,0,1) = a_1 v_1 + a_2 v_2 + a_3 v_3$

$$= a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1)$$

$$= (a_1, a_2, a_3)$$

$\Rightarrow a_1 = 1, \; a_2 = 0, \; a_3 = 1.$

$$T(e_2) = (0,1,1) = b_1 v_1 + b_2 v_2 + b_3 v_3$$

$$= b_1(1,0,0) + b_2(0,1,0) + b_3(0,0,1).$$

$$= (b_1, b_2, b_3).$$

$\Rightarrow b_1 = 0, \; b_2 = 1, \; b_3 = 1.$

$$T(e_3) = (1,1,2) = c_1 v_1 + c_2 v_2 + c_3 v_3$$

$$= c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1)$$

$$= (c_1, c_2, c_3)$$

$\Rightarrow c_1 = 1, \; c_2 = 1, \; c_3 = 2.$

$$\therefore \left.\begin{array}{l} T(e_1) = 1\cdot v_1 + 0\cdot v_2 + 1\cdot v_3 \\ T(e_2) = 0\cdot v_1 + 1\cdot v_2 + 1\cdot v_3 \\ T(e_3) = 1\cdot v_1 + 1\cdot v_2 + 2\cdot v_3 \end{array}\right\} \Rightarrow [T]_\alpha = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$[S+T]_\alpha = [S]_\alpha + [T]_\alpha = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 2 & 2 & 3 \\ 2 & 3 & 3 \end{bmatrix}.$$

$$[T \circ S]_\alpha = [S]_\alpha \cdot [T]_\alpha = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0-1 & 0+0-1 & 2+0-2 \\ 2+0+2 & 0+1+2 & 2+1+4 \\ 1+0+1 & 0+2+1 & 1+2+2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 4 & 3 & 7 \\ 2 & 3 & 5 \end{bmatrix}$$

$$[2T-S]_\alpha = 2[T]_\alpha - [S]_\alpha = 2\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 3 \\ -2 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

3) Let $S, T: R^3 \to R^3$ and $S(x,y,z) = (2x+3y,\ x+y+z,\ 2z)$
$T(x,y,z) = (y-2z,\ x+3z,\ 3x-2y)$ with α is a standard basis. Find, $[S+T]_\alpha$, $[T \circ S]_\alpha$ and $[2S-T]_\alpha$.

4) Let $T: R^3 \to R^3$ is a linear transformation defined by $T(x,y,z) = (2y+x,\ x-4y,\ 3x+y)$. Find, $[T]_\alpha$ and $[T]_\beta$ of $\alpha = \{e_1, e_2, e_3\}$ and $\beta = \{(1,1,1), (1,1,0), (1,0,0)\}$ where, e_1, e_2, e_3 are standard basis in R^3.

5) Let $T: R^3 \to R^3$ is a linear transformation such that $T(x_1, x_2, x_3) = (x_1 + x_2 + x_3,\ -x_2,\ x_1 + 4x_3)$. Find, $[T]_\alpha$ and hence $[T]_\beta$ if $\beta = \{(1,0,0), (1,1,0), (1,1,1)\}$ and α is the standard basis.