Module - 2

Laplace Transform :-

Definition:-

Let $f(t)$ be a function defined for $t \geq 0$.
The Laplace Transform of $f(t)$ is denoted by $L\{f(t)\}$ (or) $F(s)$ (or, $\bar{f}(s)$) and defined by

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

provided that integral exists. Here s is a parameter which is either real (or) Complex.

Sufficient Conditions for the existence of Laplace Transform :

1. $f(t)$ is piecewise Continuous on the interval $0 \leq t \leq a$ for any $a > 0$
2. $|f(t)| \leq K e^{at}$ for $t \geq M$, for any real Constant a and some positive Constants K and M. (This means f is of exponential order)

Laplace Transforms of some standard functions:

① $L\{0\} = 0$

② $L\{1\} = \frac{1}{s}$ $(s>0)$

③ $L\{t\} = \frac{1}{s^2}$

④ $L\{t^2\} = \frac{2!}{s^3}$

⑤ $L\{t^3\} = \frac{3!}{s^4}$

$\vdots$

⑥ $L\{t^n\} = \frac{n!}{s^{n+1}}$ $(n = 0, 1, 2, 3, \dots)$

⑦ $L\{t^{1/2}\} = \int_0^\infty e^{-st}\, t^{1/2}\, dt$

$= \frac{1}{s^{3/2}} \int_0^\infty e^{-u} \cdot u^{3/2 - 1}\, du$

Taking $st = u$, we get $dt = \frac{1}{s} du$ and $t \to 0, u \to 0$; $t \to \infty, u \to \infty$

$= \frac{\Gamma 3/2}{s^{3/2}} = \frac{1}{s^{3/2}} \cdot \frac{1}{2} \Gamma \tfrac{1}{2} = \frac{\sqrt{\pi}}{2 s^{3/2}}$

By observation, we have

$L\{t^{1/2}\} = \frac{\Gamma 3/2}{s^{3/2}}$; $L\{t^{3/2}\} = \frac{\Gamma 5/2}{s^{5/2}}$, ...

Therefore, $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$ (for $n > -1$)

(5) $L\{e^{at}\} = \dfrac{1}{s-a}$ $(s>a)$

(6) $L\{e^{-at}\} = \dfrac{1}{s+a}$ $(s+a>0)$

(7) $L\{\sin at\} = \dfrac{a}{s^2+a^2}$

(8) $L\{\cos at\} = \dfrac{s}{s^2+a^2}$

(9) $L\{\sinh at\} = \dfrac{a}{s^2-a^2}$ (Here ~~Sinhat =~~ $\sinh at = \dfrac{e^{at}-e^{-at}}{2}$)

(10) $L\{\cosh at\} = \dfrac{s}{s^2-a^2}$ (Here $\cosh at = \dfrac{e^{at}+e^{-at}}{2}$)

Properties of Laplace Transform

I. Linear property:

$$L\{af(t)+bg(t)\} = aL\{f(t)\}+bL\{g(t)\}$$

where a and b are Constants.

II. Change of scale property:

If $L\{f(t)\} = F(s)$, then

$$L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$$

III. First Shifting theorem:

If $L\{f(t)\} = F(s)$, then

(i) $L\{e^{at} f(t)\} = F(s-a)$

and (ii) $L\{e^{-at} f(t)\} = F(s+a)$

Examples

① $L\{e^{-2t} + \cos^2 t + t^2\}$

$$= L\{e^{-2t}\} + L\{\cos^2 t\} + L\{t^2\}$$

$$= \frac{1}{s+2} + L\left\{\frac{1+\cos 2t}{2}\right\} + \frac{2}{s^3}$$

$$= \frac{1}{s+2} + \frac{1}{2}\left(L(1) + L\{\cos 2t\}\right) + \frac{2}{s^3}$$

$$= \frac{1}{s+2} + \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+4}\right) + \frac{2}{s^3}$$

② $L\left\{\sqrt{t} + \frac{1}{\sqrt{t}}\right\}$ $(t>0)$

$$= L\{\sqrt{t}\} + L\left\{\frac{1}{\sqrt{t}}\right\}$$

$$= L\{t^{1/2}\} + L\{t^{-1/2}\}$$

$$= \frac{\sqrt{\pi}}{2 s^{3/2}} + \frac{\sqrt{\pi}}{s^{1/2}}.$$

③ Evaluate:

$L\{e^{t} \sin 2t\}$

Sol: Let $f(t) = \sin 2t$

Then $L\{f(t)\} = L\{\sin 2t\} = \frac{2}{s^2+4} = F(s)$

By the first shifting property, we have

$L\{e^{at} \cdot f(t)\} = F(s-a)$

Therefore, $L\{e^{t} \cdot \sin 2t\} = F(s-1)$

$$= \frac{2}{(s-1)^2+4}$$

$$= \frac{2}{s^2-2s+5}.$$

Exercise :

① $L\{\sin 2t . \cos t\}$ ② $L\{\cos^3 t\}$

③ $L\{e^{-t} . t^2\}$ ④ $L\{\sin(at+b)\}$

⑤ $L\{\sqrt{t} + \frac{1}{\sqrt{t}}\}$ ⑥ $L\{e^{2t} . \cos 3t\}$

⑦ $L\{f(t)\}$, where $f(t) = \begin{cases} \sin t, & 0<t<\pi \\ 0, & t>\pi \end{cases}$

We have

$$L\{f(t)\} = \int_0^\infty e^{-st} . f(t)\, dt$$

$$= \int_0^\pi e^{-st} . \sin t\, dt + \int_\pi^\infty e^{-st} . 0\, dt$$

$$= \frac{e^{-st}}{s^2+1^2} (s . \sin t - 1 . \cos t) \Big]_0^\pi$$

$$= \frac{e^{-\pi s}}{s^2+1} (0+1) - \frac{1}{s^2+1} (0-1)$$

$$= \frac{1}{s^2+1} (e^{-\pi s} + 1)$$