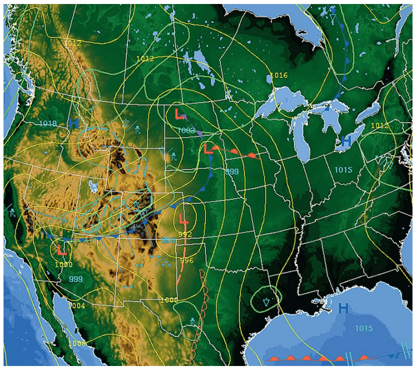
13 Functions of Several Variables



13.1 Introduction to Functions of Several Variables

Objectives

- Understand the notation for a function of several variables.
- Sketch the graph of a function of two variables.
- Sketch level curves for a function of two variables.
- Sketch level surfaces for a function of three variables.
- Use computer graphics to graph a function of two variables.

Functions of Several Variables

Functions of Several Variables

The work done by a force ($W = FD$) and the volume of a right circular cylinder ($V = \pi r^2 h$) are both functions of two variables. The volume of a rectangular solid ($V = lwh$) is a function of three variables.

The notation for a function of two or more variables is similar to that for a function of a single variable. Here are two examples.

$$z = f\underbrace{(x, y)}_{\text{2 variables}} = x^2 + xy \qquad \text{Function of two variables}$$

and

$$w = f\underbrace{(x, y, z)}_{\text{3 variables}} = x + 2y - 3z \qquad \text{Function of three variables}$$

Functions of Several Variables

DEFINITION OF A FUNCTION OF TWO VARIABLES

Let D be a set of ordered pairs of real numbers. If to each ordered pair (x, y) in D there corresponds a unique real number $f(x, y)$, then f is called a **function of x and y.** The set D is the **domain** of f, and the corresponding set of values for $f(x, y)$ is the **range** of f.

For the function given by $z = f(x, y)$, x and y are called the **independent variables** and z is called the **dependent variable.**

Example 1 – *Domains of Functions of Several Variables*

Find the domain of each function.

a. $f(x, y) = \dfrac{\sqrt{x^2 + y^2 - 9}}{x}$ **b.** $g(x, y, z) = \dfrac{x}{\sqrt{9 - x^2 - y^2 - z^2}}$

Solution:

a. The function *f* is defined for all points (*x*, *y*) such that $x \neq 0$ and $x^2 + y^2 \geq 9$.

Example 1 – *Solution*

cont'd

So, the domain is the set of all points lying on or outside the circle $x^2 + y^2 = 9$, *except* those points on the *y*-axis, as shown in Figure 13.1.

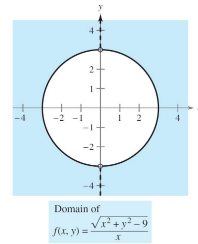


Figure 13.1

Example 1 – *Solution*

cont'd

b. The function *g* is defined for all points (*x*, *y*, *z*) such that

$$x^2 + y^2 + z^2 < 9.$$

Consequently, the domain is the set of all points (*x*, *y*, *z*) lying inside a sphere of radius 3 that is centered at the origin.

Functions of Several Variables

Functions of several variables can be combined in the same ways as functions of single variables.
For instance, you can form the sum, difference, product, and quotient of two functions of two variables as follows.

$$(f \pm g)(x, y) = f(x, y) \pm g(x, y) \qquad \text{Sum or difference}$$

$$(fg)(x, y) = f(x, y)g(x, y) \qquad \text{Product}$$

$$\frac{f}{g}(x, y) = \frac{f(x, y)}{g(x, y)}, \quad g(x, y) \neq 0 \qquad \text{Quotient}$$

Functions of Several Variables

You cannot form the composite of two functions of several variables.

However, if *h* is a function of several variables and *g* is a function of a single variable, you can form the **composite** function $(g \circ h)(x, y)$ as follows.

$$(g \circ h)(x, y) = g(h(x, y)) \qquad \text{Composition}$$

Functions of Several Variables

A function that can be written as a sum of functions of the form $cx^m y^n$ (where *c* is a real number *m* and *n* are nonnegative integers) is called a **polynomial function** of two variables.

For instance, the functions given by

$$f(x, y) = x^2 + y^2 - 2xy + x + 2 \quad \text{and} \quad g(x, y) = 3xy^2 + x - 2$$

are polynomial functions of two variables.

A **rational function** is the quotient of two polynomial functions.

Similar terminology is used for functions of more than two variables.

The Graph of a Function of Two Variables

The Graph of a Function of Two Variables

The **graph** of a function *f* of two variables is the set of all points (x, y, z) for which $z = f(x, y)$ and (x, y) is in the domain of *f*.

This graph can be interpreted geometrically as a *surface in space*. In Figure 13.2, note that the graph of $z = f(x, y)$ is a surface whose projection onto the *xy*-plane is the *D*, the domain of *f*.

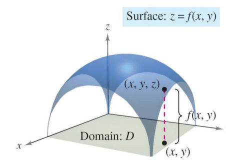


Figure 13.2

The Graph of a Function of Two Variables

To each point (x, y) in *D* there corresponds a point (x, y, z) on the surface, and, conversely, to each point (x, y, z) on the surface there corresponds a point (x, y) in *D*.

Example 2 – *Describing the Graph of a Function of Two Variables*

What is the range of $f(x, y) = \sqrt{16 - 4x^2 - y^2}$? Describe the graph of *f*.

Solution:

The domain *D* implied by the equation of *f* is the set of all points (x, y) such that $16 - 4x^2 - y^2 \geq 0$.

So, *D* is the set of all points lying on or inside the ellipse given by

$$\frac{x^2}{4} + \frac{y^2}{16} = 1. \qquad \text{Ellipse in the } xy\text{-plane}$$

Example 2 – *Solution*

cont'd

The range of *f* is all values $z = f(x, y)$ such that $0 \leq z \leq \sqrt{16}$ or

$$0 \leq z \leq 4. \qquad \text{Range of } f$$

A point (x, y, z) is on the graph of *f* if and only if

$$z = \sqrt{16 - 4x^2 - y^2}$$

$$z^2 = 16 - 4x^2 - y^2$$

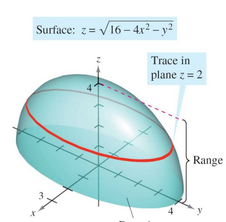
$$4x^2 + y^2 + z^2 = 16$$

$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} = 1, \qquad 0 \leq z \leq 4.$$

Example 2 – *Solution*

cont'd

You know that the graph of *f* is the upper half of an ellipsoid, as shown in Figure 13.3.



The graph of $f(x, y) = \sqrt{16 - 4x^2 - y^2}$ is the upper half of an ellipsoid.

Figure 13.3

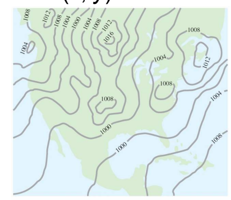
Level Curves

Level Curves

A second way to visualize a function of two variables is to use a **scalar field** in which the scalar $z = f(x, y)$ is assigned to the point (x, y).

A scalar field can be characterized by **level curves** (or **contour lines**) along which the value of $f(x, y)$ is constant.

For instance, the weather map in Figure 13.5 shows level curves of equal pressure called **isobars.**

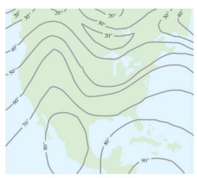


Level curves show the lines of equal pressure (isobars) measured in millibars.

Figure 13.5

Level Curves

In weather maps for which the level curves represent points of equal temperature, the level curves are called **isotherms,** as shown in Figure 13.6.



Level curves show the lines of equal temperature (isotherms) measured in degrees Fahrenheit.

Figure 13.6

Another common use of level curves is in representing electric potential fields.

In this type of map, the level curves are called **equipotential lines.**

Level Curves

Contour maps are commonly used to show regions on Earth's surface, with the level curves representing the height above sea level. This type of map is called a **topographic map.**

Level Curves

For example, the mountain shown in Figure 13.7 is represented by the topographic map in Figure 13.8.

Figure 13.7

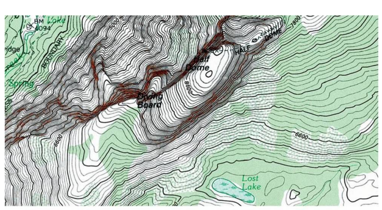


Figure 13.8

Level Curves

A contour map depicts the variation of z with respect to x and y by the spacing between level curves.

Much space between level curves indicates that z is changing slowly, whereas little space indicates a rapid change in z.

Furthermore, to produce a good three-dimensional illusion in a contour map, it is important to choose c-values that are *evenly spaced*.

Example 3 – *Sketching a Contour Map*

The hemisphere given by $f(x, y) = \sqrt{64 - x^2 - y^2}$ is shown in Figure 13.9. Sketch a contour map of this surface using level curves corresponding to c = 0, 1, 2,…, 8.

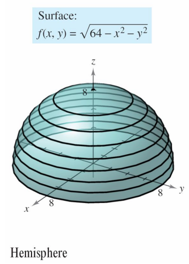


Figure 13.9

Example 3 – *Solution*

For each value of c, the equation given by $f(x, y) = c$ is a circle (or point) in the xy-plane.

For example, when c_1= 0, the level curve is

$$x^2 + y^2 = 64 \quad \text{Circle of radius 8}$$

which is a circle of radius 8.

Example 3 – *Solution*

cont'd

Figure 13.10 shows the nine level curves for the hemisphere.

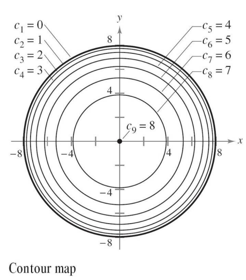


Figure 13.10

Level Curves

One example of a function of two variables used in economics is the **Cobb-Douglas production function.**

This function is used as a model to represent the numbers of units produced by varying amounts of labor and capital.

If x measures the units of labor and y measures the units of capital, the number of units produced is given by

$$f(x, y) = Cx^a y^{1-a}$$

where C and a are constants with $0 < a < 1$.

Level Surfaces

Level Surfaces

The concept of a level curve can be extended by one dimension to define a **level surface.**

If f is a function of three variables and c is a constant, the graph of the equation $f(x, y, z) = c$ is a **level surface** of the function f, as shown in Figure 13.14.

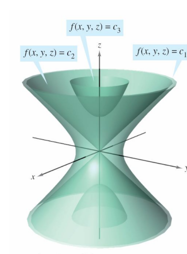
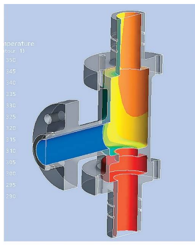


Figure 13.14

Level Surfaces

With computers, engineers and scientists have developed other ways to view functions of three variables.

For instance, Figure 13.15 shows a computer simulation that uses color to represent the temperature distribution of fluid inside a pipe fitting.



One-way coupling of ANSYS CFX™ and ANSYS Mechanical™ for thermal stress analysis

Figure 13.15

Example 6 – *Level Surfaces*

Describe the level surfaces of the function

$$f(x, y, z) = 4x^2 + y^2 + z^2.$$

Solution:

Each level surface has an equation of the form

$$4x^2 + y^2 + z^2 = c. \quad \text{Equation of level surface}$$

So, the level surfaces are ellipsoids (whose cross sections parallel to the *yz*-plane are circles).

As *c* increases, the radii of the circular cross sections increase according to the square root of *c*.

Example 6 – *Solution*

cont'd

For example, the level surfaces corresponding to the values *c* = 0, *c* = 4, *c* = 16 and are as follows.

$$4x^2 + y^2 + z^2 = 0 \quad \text{Level surface for } c = 0 \text{ (single point)}$$

$$\frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{4} = 1 \quad \text{Level surface for } c = 4 \text{ (ellipsoid)}$$

$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} = 1 \quad \text{Level surface for } c = 16 \text{ (ellipsoid)}$$

Example 6 – *Solution*

cont'd

These level surfaces are shown in Figure 13.16.

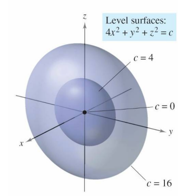


Figure 13.16

If the function represents the *temperature* at the point (*x*, *y*, *z*), the level surfaces shown in Figure 13.16 would be called **isothermal surfaces.**

Computer Graphics

Computer Graphics

The problem of sketching the graph of a surface in space can be simplified by using a computer.

Although there are several types of three-dimensional graphing utilities, most use some form of trace analysis to give the illusion of three dimensions.

To use such a graphing utility, you usually need to enter the equation of the surface, the region in the *xy*-plane over which the surface is to be plotted, and the number of traces to be taken.

Computer Graphics

For instance, to graph the surface given by

$$f(x, y) = (x^2 + y^2)e^{1-x^2-y^2}$$

you might choose the following bounds for x, y, and z.

$-3 \le x \le 3$ Bounds for x

$-3 \le y \le 3$ Bounds for y

$0 \le z \le 3$ Bounds for z

Computer Graphics

Figure 13.17 shows a computer-generated graph of this surface using 26 traces taken parallel to the *yz*-plane.

To heighten the three-dimensional effect, the program uses a "hidden line" routine.

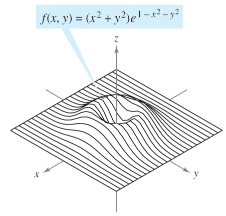


Figure 13.17

Computer Graphics

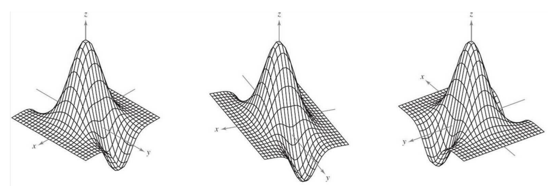
That is, it begins by plotting the traces in the foreground (those corresponding to the largest *x*-values), and then, as each new trace is plotted, the program determines whether all or only part of the next trace should be shown.

The graphs show a variety of surfaces that were plotted by computer.

If you have access to a computer drawing program, use it to reproduce these surfaces.

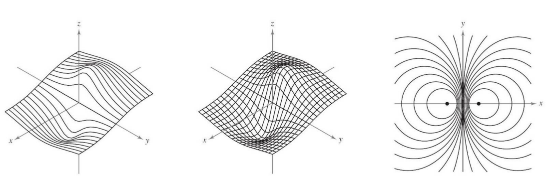
Remember also that the three-dimensional graphics in this text can be viewed and rotated.

Computer Graphics



Three different views of the graph of $f(x, y) = (2 - y^2 + x^2)e^{1-x^2-(y^2/4)}$

Computer Graphics cont'd



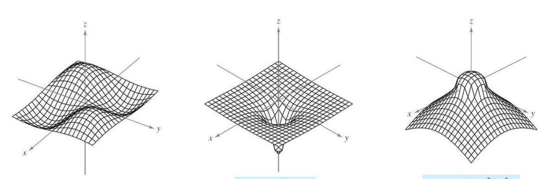
Single traces

Double traces

Level curves

Traces and level curves of the graph of $f(x, y) = \dfrac{-4x}{x^2 + y^2 + 1}$

Computer Graphics cont'd



$f(x, y) = \sin x \sin y$

$f(x, y) = -\dfrac{1}{\sqrt{x^2 + y^2}}$

$f(x, y) = \dfrac{1 - x^2 - y^2}{\sqrt{|1 - x^2 - y^2|}}$

13.2 Limits and Continuity

Objectives

- Understand the definition of a neighborhood in the plane.
- Understand and use the definition of the limit of a function of two variables.
- Extend the concept of continuity to a function of two variables.
- Extend the concept of continuity to a function of three variables.

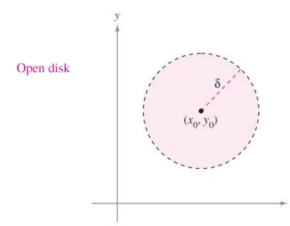
Neighborhoods in the Plane

Neighborhoods in the Plane

Using the formula for the distance between two points (x, y) and (x_0, y_0) in the plane, you can define the δ-**neighborhood** about (x_0, y_0) to be the **disk** centered at (x_0, y_0) with radius $\delta > 0$

$$\left\{(x, y): \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\right\}$$

as shown in Figure 13.18.



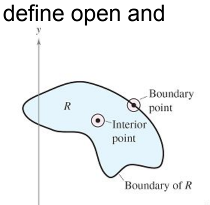
An open disk

Figure 13.18

Neighborhoods in the Plane

When this formula contains the *less than* inequality sign, <, the disk is called **open,** and when it contains the *less than or equal to* inequality sign, ≤, the disk is called **closed.** This corresponds to the use of < and ≤ to define open and closed intervals.

A point (x_0, y_0) in a plane region R is an **interior point** of R if there exists a δ-neighborhood about (x_0, y_0) that lies entirely in R, as shown in Figure 13.19.



The boundary and interior points of a region R

Figure 13.19

Neighborhoods in the Plane

If every point in R is an interior point, then R is an **open region.** A point (x_0, y_0) is a **boundary point** of R if every open disk centered at (x_0, y_0) contains points inside R *and* points outside R.

By definition, a region must contain its interior points, but it need not contain its boundary points. If a region contains all its boundary points, the region is **closed.** A region that contains some but not all of its boundary points is neither open nor closed.

Limit of a Function of Two Variables

Limit of a Function of Two Variables

DEFINITION OF THE LIMIT OF A FUNCTION OF TWO VARIABLES

Let f be a function of two variables defined, except possibly at (x_0, y_0), on an open disk centered at (x_0, y_0), and let L be a real number. Then

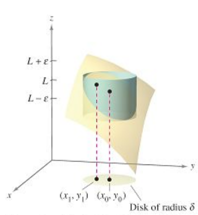
$$\lim_{(x,\,y)\to(x_0,\,y_0)} f(x, y) = L$$

if for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

Limit of a Function of Two Variables

Graphically, this definition of a limit implies that for any point $(x, y) \neq (x_0, y_0)$ in the disk of radius δ, the value $f(x, y)$ lies between $L + \varepsilon$ and $L - \varepsilon$, as shown in Figure 13.20.



For any (x, y) in the disk of radius δ, the value $f(x, y)$ lies between $L + \varepsilon$ and $L - \varepsilon$.

Figure 13.20

Limit of a Function of Two Variables

The definition of the limit of a function of two variables is similar to the definition of the limit of a function of a single variable, yet there is a critical difference.

To determine whether a function of a single variable has a limit, you need only test the approach from two directions—from the right and from the left.

If the function approaches the same limit from the right and from the left, you can conclude that the limit exists.

Limit of a Function of Two Variables

However, for a function of two variables, the statement

$(x, y) \to (x_0, y_0)$

means that the point (x, y) is allowed to approach (x_0, y_0) from any direction.

If the value of

$$\lim_{(x,\,y)\to(x_0,\,y_0)} f(x, y)$$

is not the same for all possible approaches, or **paths,** to (x_0, y_0) the limit does not exist.

Example 1 – *Verifying a Limit by the Definition*

Show that $\lim_{(x,\,y)\to(a,\,b)} x = a.$

Solution:

Let $f(x, y) = x$ and $L = a$.

You need to show that for each $\varepsilon > 0$, there exists a δ-neighborhood about (a, b) such that

$$|f(x, y) - L| = |x - a| < \varepsilon$$

whenever $(x, y) \neq (a, b)$ lies in the neighborhood.

Example 1 – *Solution*

cont'd

You can first observe that from

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

it follows that

$$\begin{aligned} |f(x, y) - a| &= |x - a| \\ &= \sqrt{(x-a)^2} \\ &\le \sqrt{(x-a)^2 + (y-b)^2} \\ &< \delta. \end{aligned}$$

So, you can choose δ = ε, and the limit is verified.

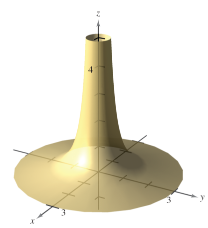
Limit of a Function of Two Variables

Limits of functions of several variables have the same properties regarding sums, differences, products, and quotients as do limits of functions of single variables.

Limit of a Function of Two Variables

For some functions, it is easy to recognize that a limit does not exist.

For instance, it is clear that the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2}$ does not exist because the values of $f(x, y)$ increase without bound as (x, y) approaches $(0, 0)$ along *any* path (see Figure 13.22).



$\displaystyle\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2}$ does not exist.

Figure 13.22

Continuity of a Function of Two Variables

Continuity of a Function of Two Variables

The limit of $f(x, y) = 5x^2y/(x^2 + y^2)$ as $(x, y) \to (1, 2)$ can be evaluated by direct substitution.

That is, the limit is $f(1, 2) = 2$.

In such cases the function *f* is said to be **continuous** at the point (1, 2).

Continuity of a Function of Two Variables

DEFINITION OF CONTINUITY OF A FUNCTION OF TWO VARIABLES

A function f of two variables is **continuous at a point** (x_0, y_0) in an open region R if $f(x_0, y_0)$ is equal to the limit of $f(x, y)$ as (x, y) approaches (x_0, y_0). That is,

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0).$$

The function f is **continuous in the open region** R if it is continuous at every point in R.

Continuity of a Function of Two Variables

The function $f(x, y) = \dfrac{5x^2y}{x^2 + y^2}$

is not continuous at (0, 0). However, because the limit at this point exists, you can remove the discontinuity by defining *f* at (0, 0) as being equal to its limit there. Such a discontinuity is called **removable.**

The function $f(x, y) = \left(\dfrac{x^2 - y^2}{x^2 + y^2}\right)^2$

is not continuous at (0, 0), and this discontinuity is **nonremovable.**

Continuity of a Function of Two Variables

THEOREM 13.1 CONTINUOUS FUNCTIONS OF TWO VARIABLES

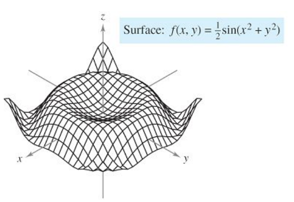
If k is a real number and f and g are continuous at (x_0, y_0), then the following functions are continuous at (x_0, y_0).

1. Scalar multiple: kf
2. Sum and difference: $f \pm g$
3. Product: fg
4. Quotient: f/g, if $g(x_0, y_0) \neq 0$

Theorem 13.1 establishes the continuity of *polynomial* and *rational* functions at every point in their domains. Furthermore, the continuity of other types of functions can be extended naturally from one to two variables.

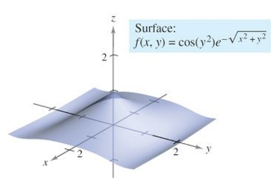
Continuity of a Function of Two Variables

For instance, the functions whose graphs are shown in Figures 13.24 and 13.25 are continuous at every point in the plane.



The function *f* is continuous at every point in the plane.

Figure 13.24



The function *f* is continuous at every point in the plane.

Figure 13.25

Continuity of a Function of Two Variables

THEOREM 13.2 CONTINUITY OF A COMPOSITE FUNCTION

If h is continuous at (x_0, y_0) and g is continuous at $h(x_0, y_0)$, then the composite function given by $(g \circ h)(x, y) = g(h(x, y))$ is continuous at (x_0, y_0). That is,

$$\lim_{(x, y)\to(x_0, y_0)} g(h(x, y)) = g(h(x_0, y_0)).$$

Example 5 – *Testing for Continuity*

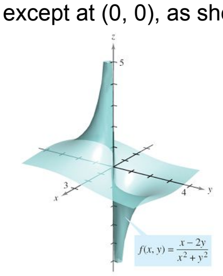
Discuss the continuity of each function.

a. $f(x, y) = \dfrac{x - 2y}{x^2 + y^2}$

b. $g(x, y) = \dfrac{2}{y - x^2}$

Example 5(a) – *Solution*

Because a rational function is continuous at every point in its domain, you can conclude that *f* is continuous at each point in the *xy*-plane except at (0, 0), as shown in Figure 13.26.



The function *f* is not continuous at (0, 0).

Figure 13.26

Example 5(b) – *Solution*

cont'd

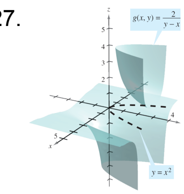
The function given by $g(x, y) = 2/(y - x^2)$ is continuous except at the points at which the denominator is 0, $y - x^2 = 0$.

So, you can conclude that the function is continuous at all points except those lying on the parabola $y = x^2$.

Example 5(b) – *Solution*

cont'd

Inside this parabola, you have $y > x^2$, and the surface represented by the function lies above the *xy*-plane, as shown in Figure 13.27.



The function g is not continuous on the parabola $y = x^2$.

Figure 13.27

Outside the parabola, $y < x^2$, and the surface lies below the *xy*-plane.

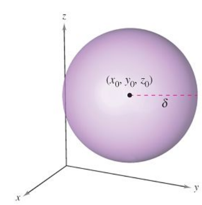
Continuity of a Function of Three Variables

Continuity of a Function of Three Variables

The definitions of limits and continuity can be extended to functions of three variables by considering points (x, y, z) within the *open sphere*

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < \delta^2. \quad \text{Open sphere}$$

The radius of this sphere is δ, and the sphere is centered at (x_0, y_0, z_0), as shown in Figure 13.28.



Open sphere in space

Figure 13.28

Continuity of a Function of Three Variables

A point (x_0, y_0, z_0) in a region R in space is an **interior point** of R if there exists a δ-sphere about (x_0, y_0, z_0) that lies entirely in R. If every point in R is an interior point, then R is called **open.**

DEFINITION OF CONTINUITY OF A FUNCTION OF THREE VARIABLES

A function f of three variables is **continuous at a point (x_0, y_0, z_0)** in an open region R if $f(x_0, y_0, z_0)$ is defined and is equal to the limit of $f(x, y, z)$ as (x, y, z) approaches (x_0, y_0, z_0). That is,

$$\lim_{(x, y, z) \to (x_0, y_0, z_0)} f(x, y, z) = f(x_0, y_0, z_0).$$

The function f is **continuous in the open region R** if it is continuous at every point in R.

Example 6 – *Testing Continuity of a Function of Three Variables*

The function

$$f(x, y, z) = \frac{1}{x^2 + y^2 - z}$$

is continuous at each point in space except at the points on the paraboloid given by $z = x^2 + y^2$.

13.3 Partial Derivatives

Objectives

- Find and use partial derivatives of a function of two variables.
- Find and use partial derivatives of a function of three or more variables.
- Find higher-order partial derivatives of a function of two or three variables.

Partial Derivatives of a Function of Two Variables

Partial Derivatives of a Function of Two Variables

You can determine the rate of change of a function *f* with respect to one of its several independent variables.

This process is called **partial differentiation,** and the result is referred to as the **partial derivative** of *f* with respect to the chosen independent variable.

Partial Derivatives of a Function of Two Variables

DEFINITION OF PARTIAL DERIVATIVES OF A FUNCTION OF TWO VARIABLES

If $z = f(x, y)$, then the **first partial derivatives** of f with respect to x and y are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

provided the limits exist.

This definition indicates that if $z = f(x, y)$, then to find f_x you *consider y constant* and differentiate with respect to x.

Similarly, to find f_y, you *consider x constant* and differentiate with respect to y.

Example 1 – *Finding Partial Derivatives*

Find the partial derivatives f_x and f_y for the function

$$f(x, y) = 3x - x^2y^2 + 2x^3y.$$

Solution:

Considering y to be constant and differentiating with respect to x produces

$f(x, y) = 3x - x^2y^2 + 2x^3y$ Write original function.

$f_x(x, y) = 3 - 2xy^2 + 6x^2y.$ Partial derivative with respect to x

Example 1 – *Solution*

cont'd

Considering x to be constant and differentiating with respect to y produces

$f(x, y) = 3x - x^2y^2 + 2x^3y$ Write original function.

$f_y(x, y) = -2x^2y + 2x^3.$ Partial derivative with respect to y

Partial Derivatives of a Function of Two Variables

NOTATION FOR FIRST PARTIAL DERIVATIVES

For $z = f(x, y)$, the partial derivatives f_x and f_y are denoted by

$$\frac{\partial}{\partial x} f(x, y) = f_x(x, y) = z_x = \frac{\partial z}{\partial x}$$

and

$$\frac{\partial}{\partial y} f(x, y) = f_y(x, y) = z_y = \frac{\partial z}{\partial y}.$$

The first partials evaluated at the point (a, b) are denoted by

$$\left.\frac{\partial z}{\partial x}\right|_{(a, b)} = f_x(a, b) \quad \text{and} \quad \left.\frac{\partial z}{\partial y}\right|_{(a, b)} = f_y(a, b).$$

Partial Derivatives of a Function of Two Variables

The partial derivatives of a function of two variables, $z = f(x, y)$, have a useful geometric interpretation. If $y = y_0$, then $z = f(x, y_0)$ represents the curve formed by intersecting the surface $z = f(x, y)$ with the plane $y = y_0$, as shown in Figure 13.29.

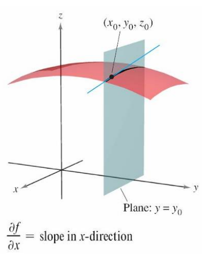


Figure 13.29

Partial Derivatives of a Function of Two Variables

Therefore,

$$f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

represents the slope of this curve at the point $(x_0, y_0, f(x_0, y_0))$.

Note that both the curve and the tangent line lie in the plane $y = y_0$.

Partial Derivatives of a Function of Two Variables

Similarly,

$$f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

represents the slope of the curve given by the intersection of $z = f(x, y)$ and the plane $x = x_0$ at $(x_0, y_0, f(x_0, y_0))$, as shown in Figure 13.30.

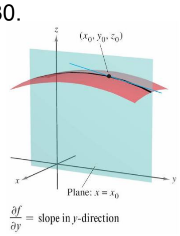


Figure 13.30

Partial Derivatives of a Function of Two Variables

Informally, the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point (x_0, y_0, z_0) denote the **slopes of the surface in the *x*- and *y*-directions,** respectively.

Example 3 – *Finding the Slopes of a Surface in the x- and y-Directions*

Find the slopes in the *x*-direction and in the *y*-direction of the surface given by

$$f(x, y) = -\frac{x^2}{2} - y^2 + \frac{25}{8}$$

at the point $\left(\frac{1}{2}, 1, 2\right)$.

Solution:

The partial derivatives of *f* with respect to *x* and *y* are

$$f_x(x, y) = -x \qquad \text{and} \qquad f_y(x, y) = -2y. \qquad \text{Partial derivatives}$$

Example 3 – *Solution*

cont'd

So, in the *x*-direction, the slope is

$$f_x\left(\frac{1}{2}, 1\right) = -\frac{1}{2} \qquad \text{Figure 13.31(a)}$$

and in the *y*-direction the slope is

$$f_y\left(\frac{1}{2}, 1\right) = -2. \qquad \text{Figure 13.31(b)}$$

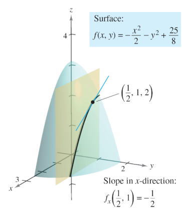


Figure 13.31(a)

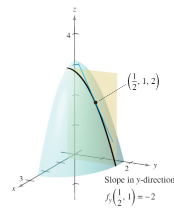


Figure 13.31(b)

Partial Derivatives of a Function of Three or More Variables

Partial Derivatives of a Function of Three or More Variables

The concept of a partial derivative can be extended naturally to functions of three or more variables. For instance, if $w = f(x, y, z)$, there are three partial derivatives, each of which is formed by holding two of the variables constant.

That is, to define the partial derivative of *w* with respect to *x*, consider *y* and *z* to be constant and differentiate with respect to *x*.

A similar process is used to find the derivatives of *w* with respect to *y* and with respect to *z*.

Partial Derivatives of a Function of Three or More Variables

$$\frac{\partial w}{\partial x} = f_x(x, y, z) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$

$$\frac{\partial w}{\partial y} = f_y(x, y, z) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$

$$\frac{\partial w}{\partial z} = f_z(x, y, z) = \lim_{\Delta z \to 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

In general, if $w = f(x_1, x_2, \ldots, x_n)$, there are *n* partial derivatives denoted by

$$\frac{\partial w}{\partial x_k} = f_{x_k}(x_1, x_2, \ldots, x_n), \quad k = 1, 2, \ldots, n.$$

To find the partial derivative with respect to one of the variables, hold the other variables constant and differentiate with respect to the given variable.

Example 6 – *Finding Partial Derivatives*

a. To find the partial derivative of $f(x, y, z) = xy + yz^2 + xz$ with respect to *z*, consider *x* and *y* to be constant and obtain

$$\frac{\partial}{\partial z}[xy + yz^2 + xz] = 2yz + x.$$

b. To find the partial derivative of $f(x, y, z) = z \sin(xy^2 + 2z)$ with respect to *z*, consider *x* and *y* to be constant. Then, using the Product Rule, you obtain

$$\begin{aligned}\frac{\partial}{\partial z}[z \sin(xy^2 + 2z)] &= (z)\frac{\partial}{\partial z}[\sin(xy^2 + 2z)] + \sin(xy^2 + 2z)\frac{\partial}{\partial z}[z] \\ &= (z)[\cos(xy^2 + 2z)](2) + \sin(xy^2 + 2z) \\ &= 2z \cos(xy^2 + 2z) + \sin(xy^2 + 2z).\end{aligned}$$

Example 6 – *Finding Partial Derivatives* cont'd

c. To find the partial derivative of $f(x, y, z, w) = (x + y + z)/w$ with respect to w, consider x, y, and z to be constant obtain

$$\frac{\partial}{\partial w}\left[\frac{x+y+z}{w}\right] = -\frac{x+y+z}{w^2}.$$

Higher-Order Partial Derivatives

Higher-Order Partial Derivatives

The function $z = f(x, y)$ has the following second partial derivatives.

1. Differentiate twice with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}.$$

2. Differentiate twice with respect to y:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}.$$

3. Differentiate first with respect to x and then with respect to y:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}.$$

Higher-Order Partial Derivatives

4. Differentiate first with respect to y and then with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}.$$

The third and fourth cases are called **mixed partial derivatives.**

Example 7 – *Finding Second Partial Derivatives*

Find the second partial derivatives of $f(x, y) = 3xy^2 – 2y + 5x^2y^2$, and determine the value of $f_{xy}(–1, 2)$.

Solution:

Begin by finding the first partial derivatives with respect to x and y.

$f_x(x, y) = 3y^2 + 10xy^2$ and $f_y(x, y) = 6xy – 2 + 10x^2y$

Then, differentiate each of these with respect to x and y.

$f_{xx}(x, y) = 10y^2$ and $f_{yy}(x, y) = 6x + 10x^2$

Example 7 – *Solution* cont'd

$f_{xy}(x, y) = 6y + 20xy$ and $f_{yx}(x, y) = 6y + 20xy$

At $(–1, 2)$ the value of f_{xy} is $f_{xy}(–1, 2) = 12 – 40 = –28$.

Higher-Order Partial Derivatives

THEOREM 13.3 EQUALITY OF MIXED PARTIAL DERIVATIVES

If f is a function of x and y such that f_{xy} and f_{yx} are continuous on an open disk R, then, for every (x, y) in R,

$$f_{xy}(x, y) = f_{yx}(x, y).$$

Example 8 – *Finding Higher-Order Partial Derivatives*

Show that $f_{xz} = f_{zx}$ and $f_{xzz} = f_{zxz} = f_{zzx}$ for the function given by

$$f(x, y, z) = ye^x + x \ln z.$$

Solution:

First partials:

$$f_x(x, y, z) = ye^x + \ln z, \qquad f_z(x, y, z) = \frac{x}{z}$$

Second partials (note that the first two are equal):

$$f_{xz}(x, y, z) = \frac{1}{z}, \qquad f_{zx}(x, y, z) = \frac{1}{z}, \qquad f_{zz}(x, y, z) = -\frac{x}{z^2}$$

Third partials (note that all three are equal):

$$f_{xzz}(x, y, z) = -\frac{1}{z^2}, \qquad f_{zxz}(x, y, z) = -\frac{1}{z^2}, \qquad f_{zzx}(x, y, z) = -\frac{1}{z^2}$$

13.4

Differentials

Objectives

- Understand the concepts of increments and differentials.
- Extend the concept of differentiability to a function of two variables.
- Use a differential as an approximation.

Increments and Differentials

Increments and Differentials

For $y = f(x)$, the differential of y was defined as

$dy = f'(x)dx$.

Similar terminology is used for a function of two variables, $z = f(x, y)$. That is, Δx and Δy are the **increments of x and y**, and the **increment of z** is given by

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y). \qquad \text{Increment of } z$$

DEFINITION OF TOTAL DIFFERENTIAL

If $z = f(x, y)$ and Δx and Δy are increments of x and y, then the **differentials** of the independent variables x and y are

$$dx = \Delta x \qquad \text{and} \qquad dy = \Delta y$$

and the **total differential** of the dependent variable z is

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = f_x(x, y)\,dx + f_y(x, y)\,dy.$$

This definition can be extended to a function of three or more variables.

For instance, if $w = f(x, y, z, u)$, then $dx = \Delta x$, $dy = \Delta y$, $dz = \Delta z$, $du = \Delta u$, and the total differential of w is

$$dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz + \frac{\partial w}{\partial u}\,du.$$

Example 1 – *Finding the Total Differential*

Find the total differential for each function.

a. $z = 2x \sin y - 3x^2y^2$ **b.** $w = x^2 + y^2 + z^2$

Solution:

a. The total differential dz for $z = 2x \sin y - 3x^2y^2$ is

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy \qquad \text{Total differential } dz$$

$$= (2 \sin y - 6xy^2)dx + (2x \cos y - 6x^2y)dy.$$

Example 1– *Solution*

cont'd

b. The total differential dw for $w = x^2 + y^2 + z^2$ is

$$dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz \qquad \text{Total differential } dw$$

$$= 2x\,dx + 2y\,dy + 2z\,dz.$$

Differentiability

Differentiability

For a *differentiable* function given by $y = f(x)$, you can use the differential $dy = f'(x)dx$ as an approximation (for small Δx) to the value $\Delta y = f(x + \Delta x) - f(x)$.

When a similar approximation is possible for a function of two variables, the function is said to be **differentiable.**

Differentiability

This is stated explicitly in the following definition.

DEFINITION OF DIFFERENTIABILITY

A function f given by $z = f(x, y)$ is **differentiable** at (x_0, y_0) if Δz can be written in the form

$$\Delta z = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where both ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. The function f is **differentiable in a region *R*** if it is differentiable at each point in R.

Example 2 – *Showing That a Function Is Differentiable*

Show that the function given by

$f(x, y) = x^2 + 3y$

is differentiable at every point in the plane.

Solution:

Letting $z = f(x, y)$, the increment of z at an arbitrary point (x, y) in the plane is

$$\begin{aligned} \Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y) \quad \text{Increment of } z \\ &= (x^2 + 2x\Delta x + \Delta x^2) + 3(y + \Delta y) - (x^2 + 3y) \\ &= 2x\Delta x + \Delta x^2 + 3\Delta y \end{aligned}$$

Example 2 – *Solution*

cont'd

$$\begin{aligned} &= 2x(\Delta x) + 3(\Delta y) + \Delta x(\Delta x) + 0(\Delta y) \\ &= f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \end{aligned}$$

where $\varepsilon_1 = \Delta x$ and $\varepsilon_2 = 0$.

Because $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$, it follows that f is differentiable at every point in the plane.

The graph of f is shown in Figure 13.34.

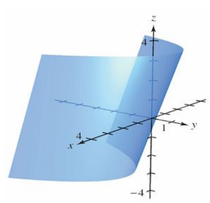


Figure 13.34

Differentiability

THEOREM 13.4 SUFFICIENT CONDITION FOR DIFFERENTIABILITY

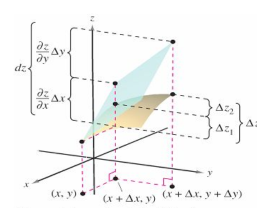
If f is a function of x and y, where f_x and f_y are continuous in an open region R, then f is differentiable on R.

Approximation by Differentials

Approximation by Differentials

Theorem 13.4 tells you that you can choose $(x + \Delta x, y + \Delta y)$ close enough to (x, y) to make $\varepsilon_1 \Delta x$ and $\varepsilon_2 \Delta y$ insignificant. In other words, for small Δx and Δy, you can use the approximation $\Delta z \approx dz$.

This approximation is illustrated graphically in Figure 13.35



The exact change in z is Δz. This change can be approximated by the differential dz.

Figure 13.35

Approximation by Differentials

The partial derivatives $\partial z/\partial x$ *and* $\partial z/\partial y$ can be interpreted as the slopes of the surface in the *x*- and *y*-directions.

This means that

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

represents the change in height of a plane that is tangent to the surface at the point $(x, y, f(x, y))$.

Because a plane in space is represented by a linear equation in the variables *x*, *y,* and *z*, the approximation of Δz by dz is called a **linear approximation.**

Example 3 – *Using a Differential as an Approximation*

Use the differential dz to approximate the change in $z = \sqrt{4 - x^2 - y^2}$ as (x, y) moves from the point (1, 1) to the point (1.01, 0.97). Compare this approximation with the exact change in *z*.

Example 3 – *Solution*

Letting $(x, y) = (1, 1)$ and $(x + \Delta x, y + \Delta y) = (1.01, 0.97)$ produces $dx = \Delta x = 0.01$ and $dy = \Delta y = -0.03$.

So, the change in *z* can be approximated by

$$\Delta z \approx dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{-x}{\sqrt{4 - x^2 - y^2}}\Delta x + \frac{-y}{\sqrt{4 - x^2 - y^2}}\Delta y.$$

When $x = 1$ and $y = 1$, you have

$$\Delta z \approx -\frac{1}{\sqrt{2}}(0.01) - \frac{1}{\sqrt{2}}(-0.03) = \frac{0.02}{\sqrt{2}} = \sqrt{2}\,(0.01) \approx 0.0141.$$

Example 3 – *Solution*

cont'd

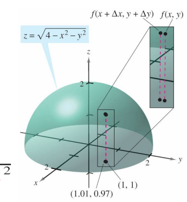
In Figure 13.36, you can see that the exact change corresponds to the difference in the heights of two points on the surface of a hemisphere.

This difference is given by

$$\Delta z = f(1.01, 0.97) - f(1, 1)$$

$$= \sqrt{4 - (1.01)^2 - (0.97)^2} - \sqrt{4 - 1^2 - 1^2}$$

$$\approx 0.0137$$



As (x, y) moves from (1, 1) to the point (1.01, 0.97), the value of $f(x, y)$ changes by about 0.0137.

Figure 13.36

Approximation by Differentials

A function of three variables $w = f(x, y, z)$ is called **differentiable** at (x, y, z) provided that

$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$

can be written in the form

$\Delta w = f_x\Delta x + f_y\Delta y + f_z\Delta z + \varepsilon_1\Delta x + \varepsilon_2\Delta y + \varepsilon_3\Delta z$

where ε_1, ε_2, and $\varepsilon_3 \to 0$ as $(\Delta x, \Delta y, \Delta z) \to (0, 0, 0)$.

With this definition of differentiability, Theorem 13.4 has the following extension for functions of three variables: If *f* is a function of *x*, *y*, and *z*, where f, f_x, f_y, and f_z are continuous in an open region *R*, then *f* is differentiable on *R*.

Approximation by Differentials

THEOREM 13.5 DIFFERENTIABILITY IMPLIES CONTINUITY

If a function of x and y is differentiable at (x_0, y_0), then it is continuous at (x_0, y_0).

Example 5 – *A Function That Is Not Differentiable*

Show that $f_x(0, 0)$ and $f_y(0, 0)$ both exist, but that f is not differentiable at (0, 0) where f is defined as

$$f(x, y) = \begin{cases} \dfrac{-3xy}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}.$$

Example 5 – *Solution*

You can show that f is not differentiable at (0, 0) by showing that it is not continuous at this point.

To see that f is not continuous at (0, 0), look at the values of $f(x, y)$ along two different approaches to (0, 0), as shown in Figure 13.38.

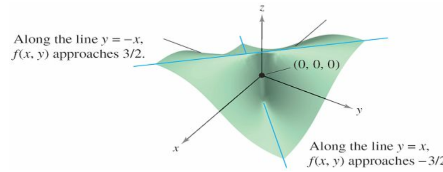


Figure 13.38

Example 5 – *Solution*

cont'd

Along the line $y = x$, the limit is

$$\lim_{(x, x)\to(0, 0)} f(x, y) = \lim_{(x, x)\to(0, 0)} \frac{-3x^2}{2x^2} = -\frac{3}{2}$$

whereas along $y = -x$ you have

$$\lim_{(x, -x)\to(0, 0)} f(x, y) = \lim_{(x, -x)\to(0, 0)} \frac{3x^2}{2x^2} = \frac{3}{2}.$$

So, the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ does not exist, and you can conclude that f is not continuous at (0, 0).

Therefore, by Theorem 13.5, you know that f is not differentiable at (0, 0).

Example 5 – *Solution*

cont'd

On the other hand, by the definition of the partial derivatives f_x and f_y you have

$$f_x(0, 0) = \lim_{\Delta x\to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{0 - 0}{\Delta x} = 0$$

and

$$f_y(0, 0) = \lim_{\Delta y\to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y\to 0} \frac{0 - 0}{\Delta y} = 0.$$

So, the partial derivatives at (0, 0) exist.

$$f(x, y) = \begin{cases} \dfrac{-3xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

13.5 Chain Rules for Functions of Several Variables

Objectives

- Use the Chain Rules for functions of several variables.
- Find partial derivatives implicitly.

Chain Rules for Functions of Several Variables

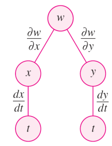
Chain Rules for Functions of Several Variables

THEOREM 13.6 CHAIN RULE: ONE INDEPENDENT VARIABLE

Let $w = f(x, y)$, where f is a differentiable function of x and y. If $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of t, then w is a differentiable function of t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

See Figure 13.39.



Chain Rule: one independent variable w is a function of x and y, which are each functions of t. This diagram represents the derivative of w with respect to t.

Figure 13.39

Example 1 – *Using the Chain Rule with One Independent Variable*

Let $w = x^2y - y^2$, where $x = \sin t$ and $y = e^t$. Find dw/dt when $t = 0$.

Solution:

By the Chain Rule for one independent variable, you have

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= 2xy(\cos t) + (x^2 - 2y)e^t$$

Example 1 – *Solution*

cont'd

$$= 2(\sin t)(e^t)(\cos t) + (\sin^2 t - 2e^t)e^t$$

$$= 2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t}.$$

When $t = 0$, it follows that

$$\frac{dw}{dt} = -2.$$

Chain Rules for Functions of Several Variables

The Chain Rule in Theorem 13.6 can be extended to any number of variables. For example, if each x_i is a differentiable function of a single variable *t,* then for

$w = f(x_1, x_2, \ldots, x_n)$

you have

$$\frac{dw}{dt} = \frac{\partial w}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial w}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial w}{\partial x_n}\frac{dx_n}{dt}.$$

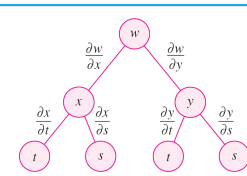
Chain Rules for Functions of Several Variables

THEOREM 13.7 CHAIN RULE: TWO INDEPENDENT VARIABLES

Let $w = f(x, y)$, where f is a differentiable function of x and y. If $x = g(s, t)$ and $y = h(s, t)$ such that the first partials $\partial x/\partial s$, $\partial x/\partial t$, $\partial y/\partial s$, and $\partial y/\partial t$ all exist, then $\partial w/\partial s$ and $\partial w/\partial t$ exist and are given by

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}.$$

The Chain Rule in Theorem 13.7 is shown schematically in Figure 13.41.



Chain Rule: two independent variables

Figure 13.41

Example 4 – *The Chain Rule with Two Independent Variables*

Use the Chain Rule to find $\partial w/\partial s$ and $\partial w/\partial t$ for

$w = 2xy$

where $x = s^2 + t^2$ and $y = s/t$.

Solution:

Using Theorem 13.7, you can hold t constant and differentiate with respect to s to obtain

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

$$= 2y(2s) + 2x\left(\frac{1}{t}\right)$$

Example 4 – *Solution*

cont'd

$$= 2\left(\frac{s}{t}\right)(2s) + 2(s^2 + t^2)\left(\frac{1}{t}\right) \quad \text{Substitute } (s/t) \text{ for } y \text{ and } s^2 + t^2 \text{ for } x.$$

$$= \frac{4s^2}{t} + \frac{2s^2 + 2t^2}{t}$$

$$= \frac{6s^2 + 2t^2}{t}.$$

Similarly, holding s constant gives

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$

Example 4 – *Solution*

cont'd

$$= 2y(2t) + 2x\left(\frac{-s}{t^2}\right)$$

$$= 2\left(\frac{s}{t}\right)(2t) + 2(s^2 + t^2)\left(\frac{-s}{t^2}\right) \quad \text{Substitute } (s/t) \text{ for } y \text{ and } s^2 + t^2 \text{ for } x.$$

$$= 4s - \frac{2s^3 + 2st^2}{t^2}$$

$$= \frac{4st^2 - 2s^3 - 2st^2}{t^2}$$

$$= \frac{2st^2 - 2s^3}{t^2}.$$

Chain Rules for Functions of Several Variables

The Chain Rule in Theorem 13.7 can also be extended to any number of variables. For example, if w is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$, where each x_i is a differentiable function of m the variables $t_1, t_2, \ldots, t_m$, then for $w = f(x_1, x_2, \ldots, x_n)$ you obtain the following.

$$\frac{\partial w}{\partial t_1} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_1}$$

$$\frac{\partial w}{\partial t_2} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_2} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_2} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_2}$$

$$\vdots$$

$$\frac{\partial w}{\partial t_m} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_m} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_m} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_m}$$

Implicit Partial Differentiation

Implicit Partial Differentiation

An application of the Chain Rule to determine the derivative of a function defined *implicitly*.

Suppose that x and y are related by the equation $F(x, y) = 0$, where it is assumed that $y = f(x)$ is a differentiable function of x. To find dy/dx use chain rule.

If you consider the function given by

$w = F(x, y) = F(x, f(x))$

you can apply Theorem 13.6 to obtain

$$\frac{dw}{dx} = F_x(x, y)\frac{dx}{dx} + F_y(x, y)\frac{dy}{dx}.$$

Implicit Partial Differentiation

Because $w = F(x, y) = 0$ for all x in the domain of f, you know that $dw/dx = 0$ and you have

$$F_x(x, y)\frac{dx}{dx} + F_y(x, y)\frac{dy}{dx} = 0.$$

Now, if $F_y(x, y) \neq 0$, you can use the fact that $dx/dx = 1$ to conclude that

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}.$$

A similar procedure can be used to find the partial derivatives of functions of several variables that are defined implicitly.

Implicit Partial Differentiation

THEOREM 13.8 CHAIN RULE: IMPLICIT DIFFERENTIATION

If the equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x, then

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}, \qquad F_y(x, y) \neq 0.$$

If the equation $F(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}, \quad F_z(x, y, z) \neq 0.$$

Example 6 – *Finding a Derivative Implicitly*

Find dy/dx, given $y^3 + y^2 - 5y - x^2 + 4 = 0$.

Solution:

Begin by defining a function F as

$F(x, y) = y^3 + y^2 - 5y - x^2 + 4.$

Then, using Theorem 13.8, you have

$F_x(x, y) = -2x$ and $F_y(x, y) = 3y^2 + 2y - 5$

and it follows that

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} = \frac{-(-2x)}{3y^2 + 2y - 5} = \frac{2x}{3y^2 + 2y - 5}.$$

13.6 Directional Derivatives and Gradients

Objectives

- Find and use directional derivatives of a function of two variables.
- Find the gradient of a function of two variables.
- Use the gradient of a function of two variables in applications.
- Find directional derivatives and gradients of functions of three variables.

Directional Derivative

Directional Derivative

You are standing on the hillside pictured in Figure 13.42 and want to determine the hill's incline toward the z-axis.

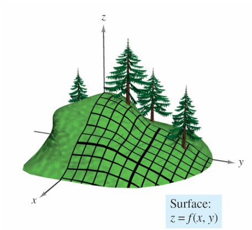


Figure 13.42

Directional Derivative

If the hill were represented by $z = f(x, y)$, you would already know how to determine the slopes in two different directions—the slope in the y-direction would be given by the partial derivative $f_y(x, y)$, and the slope in the x-direction would be given by the partial derivative $f_x(x, y)$.

In this section, you will see that these two partial derivatives can be used to find the slope in *any* direction.

Directional Derivative

To determine the slope at a point on a surface, you will define a new type of derivative called a **directional derivative.**

Begin by letting $z = f(x, y)$ be a *surface* and $P(x_0, y_0)$ be a *point* in the domain of f, as shown in Figure 13.43.

The "direction" of the directional derivative is given by a unit vector

$$\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

where θ is the angle the vector makes with the positive x-axis.

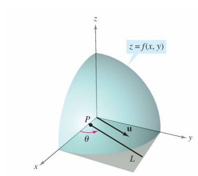


Figure 13.43

Directional Derivative

To find the desired slope, reduce the problem to two dimensions by intersecting the surface with a vertical plane passing through the point P and parallel to $\mathbf{u}$, as shown in Figure 13.44.

This vertical plane intersects the surface to form a curve C.

The slope of the surface at $(x_0, y_0, f(x_0, y_0))$ in the direction of $\mathbf{u}$ is defined as the slope of the curve C at that point.

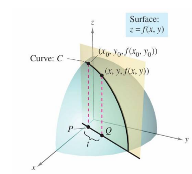


Figure 13.44

Directional Derivative

You can write the slope of the curve C as a limit that looks much like those used in single-variable calculus.

The vertical plane used to form C intersects the xy-plane in a line L, represented by the parametric equations

$$x = x_0 + t\cos\theta \qquad \text{and} \qquad y = y_0 + t\sin\theta$$

so that for any value of t, the point $Q(x, y)$ lies on the line L.

For each of the points P and Q, there is a corresponding point on the surface.

$(x_0, y_0, f(x_0, y_0))$ Point above P

$(x, y, f(x, y))$ Point above Q

Directional Derivative

Moreover, because the distance between P and Q is

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = \sqrt{(t\cos\theta)^2 + (t\sin\theta)^2} = |t|$$

you can write the slope of the secant line through $(x_0, y_0, f(x_0, y_0))$ and $(x, y, f(x, y))$ as

$$\frac{f(x, y) - f(x_0, y_0)}{t} = \frac{f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0, y_0)}{t}.$$

Finally, by letting t approach 0, you arrive at the following definition.

Directional Derivative

DEFINITION OF DIRECTIONAL DERIVATIVE

Let f be a function of two variables x and y and let $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ be a unit vector. Then the **directional derivative of f in the direction of u,** denoted by $D_{\mathbf{u}}f$, is

$$D_{\mathbf{u}}f(x, y) = \lim_{t\to 0} \frac{f(x + t\cos\theta, y + t\sin\theta) - f(x, y)}{t}$$

provided this limit exists.

THEOREM 13.9 DIRECTIONAL DERIVATIVE

If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ is

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta.$$

Directional Derivative

There are infinitely many directional derivatives of a surface at a given point—one for each direction specified by **u**, as shown in Figure 13.45.

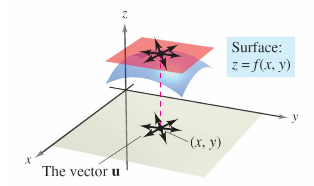


Figure 13.45

Directional Derivative

Two of these are the partial derivatives f_x and f_y.

1. Direction of positive x-axis ($\theta = 0$): $\mathbf{u} = \cos 0\,\mathbf{i} + \sin 0\,\mathbf{j} = \mathbf{i}$

$$D_{\mathbf{i}}f(x, y) = f_x(x, y)\cos 0 + f_y(x, y)\sin 0 = f_x(x, y)$$

2. Direction of positive y-axis ($\theta = \pi/2$): $\mathbf{u} = \cos\frac{\pi}{2}\,\mathbf{i} + \sin\frac{\pi}{2}\,\mathbf{j} = \mathbf{j}$

$$D_{\mathbf{j}}f(x, y) = f_x(x, y)\cos\frac{\pi}{2} + f_y(x, y)\sin\frac{\pi}{2} = f_y(x, y)$$

Example 1 – *Finding a Directional Derivative*

Find the directional derivative of

$$f(x, y) = 4 - x^2 - \tfrac{1}{4}y^2 \qquad \text{Surface}$$

at (1, 2) in the direction of

$$\mathbf{u} = \left(\cos\frac{\pi}{3}\right)\mathbf{i} + \left(\sin\frac{\pi}{3}\right)\mathbf{j}. \qquad \text{Direction}$$

Example 1 – *Solution*

Because f_x and f_y are continuous, f is differentiable, and you can apply Theorem 13.9.

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$
$$= (-2x)\cos\theta + \left(-\frac{y}{2}\right)\sin\theta$$

Evaluating at $\theta = \pi/3$, $x = 1$, and $y = 2$ produces

$$D_{\mathbf{u}}f(1, 2) = (-2)\left(\frac{1}{2}\right) + (-1)\left(\frac{\sqrt{3}}{2}\right)$$
$$= -1 - \frac{\sqrt{3}}{2}$$
$$\approx -1.866. \qquad \text{See Figure 13.46.}$$

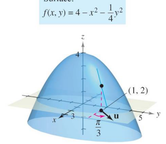


Figure 13.46

The Gradient of a Function of Two Variables

The Gradient of a Function of Two Variables

The **gradient** of a function of two variables is a vector-valued function of two variables.

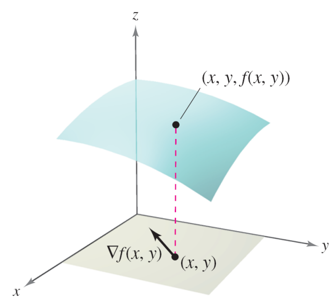
DEFINITION OF GRADIENT OF A FUNCTION OF TWO VARIABLES

Let $z = f(x, y)$ be a function of x and y such that f_x and f_y exist. Then the **gradient of *f*,** denoted by $\nabla f(x, y)$, is the vector

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}.$$

∇f is read as "del *f*." Another notation for the gradient is **grad** $f(x, y)$. In Figure 13.48, note that for each (x, y), the gradient $\nabla f(x, y)$ is a vector in the plane (not a vector in space).

The Gradient of a Function of Two Variables



The gradient of f is a vector in the xy-plane.

Figure 13.48

Example 3 – *Finding the Gradient of a Function*

Find the gradient of $f(x, y) = y \ln x + xy^2$ at the point (1, 2).

Solution:

Using

$$f_x(x, y) = \frac{y}{x} + y^2 \quad \text{and} \quad f_y(x, y) = \ln x + 2xy$$

you have

$$\nabla f(x, y) = \left(\frac{y}{x} + y^2\right)\mathbf{i} + (\ln x + 2xy)\mathbf{j}.$$

At the point (1, 2), the gradient is

$$\nabla f(1, 2) = \left(\frac{2}{1} + 2^2\right)\mathbf{i} + [\ln 1 + 2(1)(2)]\mathbf{j}$$

$$= 6\mathbf{i} + 4\mathbf{j}.$$

The Gradient of a Function of Two Variables

THEOREM 13.10 ALTERNATIVE FORM OF THE DIRECTIONAL DERIVATIVE

If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

Example 4 – *Using ∇f(x, y) to Find a Directional Derivative*

Find the directional derivative of

$$f(x, y) = 3x^2 - 2y^2$$

at $\left(-\frac{3}{4}, 0\right)$ in the direction from $P\left(-\frac{3}{4}, 0\right)$ to $Q(0, 1)$.

Solution:

Because the partials of f are continuous, f is differentiable and you can apply Theorem 13.10.

A vector in the specified direction is

$$\overrightarrow{PQ} = \mathbf{v} = \left(0 + \frac{3}{4}\right)\mathbf{i} + (1 - 0)\mathbf{j}$$

Example 4 – *Solution*

cont'd

$$= \frac{3}{4}\mathbf{i} + \mathbf{j}$$

and a unit vector in this direction is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}. \qquad \text{Unit vector in direction of } \overrightarrow{PQ}$$

Because $\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = 6x\mathbf{i} - 4y\mathbf{j}$, the gradient at $\left(-\frac{3}{4}, 0\right)$ is

$$\nabla f\left(-\frac{3}{4}, 0\right) = -\frac{9}{2}\mathbf{i} + 0\mathbf{j}. \qquad \text{Gradient at } \left(-\tfrac{3}{4}, 0\right)$$

Example 4 – *Solution*

cont'd

Consequently, at $\left(-\frac{3}{4}, 0\right)$ the directional derivative is

$$D_{\mathbf{u}} f\left(-\frac{3}{4}, 0\right) = \nabla f\left(-\frac{3}{4}, 0\right) \cdot \mathbf{u}$$

$$= \left(-\frac{9}{2}\mathbf{i} + 0\mathbf{j}\right) \cdot \left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right)$$

$$= -\frac{27}{10}. \quad \text{Directional derivative at } \left(-\tfrac{3}{4}, 0\right)$$

See Figure 13.49.

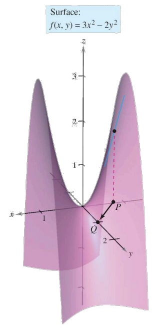


Figure 13.49

Applications of the Gradient

Applications of the Gradient

THEOREM 13.11 PROPERTIES OF THE GRADIENT

Let f be differentiable at the point (x, y).

1. If $\nabla f(x, y) = \mathbf{0}$, then $D_{\mathbf{u}} f(x, y) = 0$ for all $\mathbf{u}$.
2. The direction of *maximum* increase of f is given by $\nabla f(x, y)$. The maximum value of $D_{\mathbf{u}} f(x, y)$ is $\|\nabla f(x, y)\|$.
3. The direction of *minimum* increase of f is given by $-\nabla f(x, y)$. The minimum value of $D_{\mathbf{u}} f(x, y)$ is $-\|\nabla f(x, y)\|$.

Example 5 – *Finding the Direction of Maximum Increase*

The temperature in degrees Celsius on the surface of a metal plate is

$$T(x, y) = 20 - 4x^2 - y^2$$

where x and y are measured in centimeters. In what direction from $(2, -3)$ does the temperature increase most rapidly? What is this rate of increase?

Solution:

The gradient is

$$\nabla T(x, y) = T_x(x, y)\mathbf{i} + T_y(x, y)\mathbf{j}$$

$$= -8x\mathbf{i} - 2y\mathbf{j}.$$

Example 5 – *Solution*

cont'd

It follows that the direction of maximum increase is given by

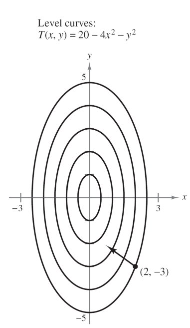
$$\nabla T(2, -3) = -16\mathbf{i} + 6\mathbf{j}$$

as shown in Figure 13.51, and the rate of increase is

$$\|\nabla T(2, -3)\| = \sqrt{256 + 36}$$

$$= \sqrt{292}$$

$$\approx 17.09° \text{ per centimeter.}$$



The direction of most rapid increase in temperature at $(2, -3)$ is given by $-16\mathbf{i} + 6\mathbf{j}$.

Figure 13.51

Applications of the Gradient

THEOREM 13.12 GRADIENT IS NORMAL TO LEVEL CURVES

If f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0).

Example 7 – *Finding a Normal Vector to a Level Curve*

Sketch the level curve corresponding to $c = 0$ for the function given by $f(x, y) = y - \sin x$ and find a normal vector at several points on the curve.

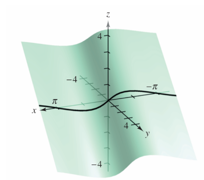
Solution:

The level curve for $c = 0$ is given by

$0 = y - \sin x$

$y = \sin x$

as shown in Figure 13.53(a).



The surface is given by $f(x, y) = y - \sin x$.

Figure 13.53(a)

Example 7 – *Solution*

cont'd

Because the gradient vector of f at (x, y) is

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = -\cos x\mathbf{i} + \mathbf{j}$$

you can use Theorem 13.12 to conclude that $\nabla f(x, y)$ is normal to the level curve at the point (x, y).

Some gradient vectors are

$$\nabla f(-\pi, 0) = \mathbf{i} + \mathbf{j}$$

$$\nabla f\left(-\frac{2\pi}{3}, -\frac{\sqrt{3}}{2}\right) = \frac{1}{2}\mathbf{i} + \mathbf{j}$$

$$\nabla f\left(-\frac{\pi}{2}, -1\right) = \mathbf{j}$$

Example 7 – *Solution*

cont'd

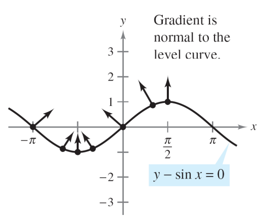
$$\nabla f\left(-\frac{\pi}{3}, -\frac{\sqrt{3}}{2}\right) = -\frac{1}{2}\mathbf{i} + \mathbf{j}$$

$$\nabla f(0, 0) = -\mathbf{i} + \mathbf{j}$$

$$\nabla f\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right) = -\frac{1}{2}\mathbf{i} + \mathbf{j}$$

$$\nabla f\left(\frac{\pi}{2}, 1\right) = \mathbf{j}.$$

These are shown in Figure 13.53(b).



The level curve is given by $f(x, y) = 0$.

Figure 13.53(b)

Functions of Three Variables

Functions of Three Variables

DIRECTIONAL DERIVATIVE AND GRADIENT FOR THREE VARIABLES

Let f be a function of x, y, and z, with continuous first partial derivatives. The **directional derivative of f** in the direction of a unit vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is given by

$$D_{\mathbf{u}} f(x, y, z) = af_x(x, y, z) + bf_y(x, y, z) + cf_z(x, y, z).$$

The **gradient of f** is defined as

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}.$$

Properties of the gradient are as follows.

1. $D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$
2. If $\nabla f(x, y, z) = \mathbf{0}$, then $D_{\mathbf{u}} f(x, y, z) = 0$ for all $\mathbf{u}$.
3. The direction of *maximum* increase of f is given by $\nabla f(x, y, z)$. The maximum value of $D_{\mathbf{u}} f(x, y, z)$ is

   $\|\nabla f(x, y, z)\|$. Maximum value of $D_{\mathbf{u}} f(x, y, z)$
4. The direction of *minimum* increase of f is given by $-\nabla f(x, y, z)$. The minimum value of $D_{\mathbf{u}} f(x, y, z)$ is

   $-\|\nabla f(x, y, z)\|$. Minimum value of $D_{\mathbf{u}} f(x, y, z)$

Example 8 – *Finding the Gradient for a Function of Three Variables*

Find $\nabla f(x, y, z)$ for the function given by

$$f(x, y, z) = x^2 + y^2 - 4z$$

and find the direction of maximum increase of f at the point $(2, -1, 1)$.

Solution:

The gradient vector is given by

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

$$= 2x\mathbf{i} + 2y\mathbf{j} - 4\mathbf{k}$$

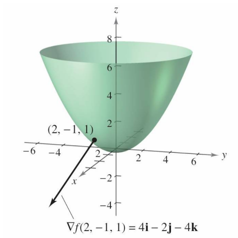
Example 8 – *Solution*

cont'd

So, it follows that the direction of maximum increase at (2, –1, 1) is

$\nabla f(2, -1, 1) = 4\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$.

See Figure 13.54.



Level surface and gradient vector at (2, −1, 1) for $f(x, y, z) = x^2 + y^2 - 4z$

Figure 13.54

13.7 Tangent Planes and Normal Lines

Objectives

- Find equations of tangent planes and normal lines to surfaces.
- Find the angle of inclination of a plane in space.
- Compare the gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$.

Tangent Plane and Normal Line to a Surface

Tangent Plane and Normal Line to a Surface

You can represent the surfaces in space primarily by equations of the form

$z = f(x, y)$. Equation of a surface S

In the development to follow, however, it is convenient to use the more general representation $F(x, y, z) = 0$.

For a surface S given by $z = f(x, y)$, you can convert to the general form by defining F as

$$F(x, y, z) = f(x, y) - z.$$

Tangent Plane and Normal Line to a Surface

Because $f(x, y) - z = 0$, you can consider S to be the level surface of F given by

$F(x, y, z) = 0$. Alternative equation of surface S

Example 1 – *Writing an Equation of a Surface*

For the function given by

$$F(x, y, z) = x^2 + y^2 + z^2 - 4$$

describe the level surface given by $F(x, y, z) = 0$.

Solution:

The level surface given by $F(x, y, z) = 0$ can be written as

$$x^2 + y^2 + z^2 = 4$$

which is a sphere of radius 2 whose center is at the origin.

Tangent Plane and Normal Line to a Surface

Normal lines are important in analyzing surfaces and solids. For example, consider the collision of two billiard balls.

When a stationary ball is struck at a point P on its surface, it moves along the **line of impact** determined by P and the center of the ball.

The impact can occur in *two* ways. If the cue ball is moving along the line of impact, it stops dead and imparts all of its momentum to the stationary ball, as shown in Figure 13.55.

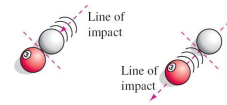


Figure 13.55

Tangent Plane and Normal Line to a Surface cont'd

If the cue ball is not moving along the line of impact, it is deflected to one side or the other and retains part of its momentum.

That part of the momentum that is transferred to the stationary ball occurs along the line of impact, *regardless* of the direction of the cue ball, as shown in Figure 13.56.

This line of impact is called the **normal line** to the surface of the ball at the point P.

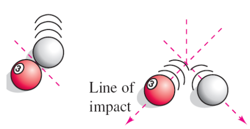


Figure 13.56

Tangent Plane and Normal Line to a Surface

In the process of finding a normal line to a surface, you are also able to solve the problem of finding a **tangent plane** to the surface.

Let S be a surface given by

$$F(x, y, z) = 0$$

and let $P(x_0, y_0, z_0)$ be a point on S.

Let C be a curve on S through P that is defined by the vector-valued function

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

Then, for all t,

$$F(x(t), y(t), z(t)) = 0.$$

Tangent Plane and Normal Line to a Surface

If F is differentiable and $x'(t)$, $y'(t)$, and $z'(t)$ all exist, it follows from the Chain Rule that

$$0 = F'(t)$$

$$= F_x(x, y, z)x'(t) + F_y(x, y, z)y'(t) + F_z(x, y, z)z'(t)$$

At (x_0, y_0, z_0), the equivalent vector form is

$$0 = \underbrace{\nabla F(x_0, y_0, z_0)}_{\text{Gradient}} \cdot \underbrace{\mathbf{r}'(t_0)}_{\text{Tangent vector}}.$$

Tangent Plane and Normal Line to a Surface

This result means that the gradient at P is orthogonal to the tangent vector of every curve on S through P. So, all tangent lines on S lie in a plane that is normal to $\nabla F(x_0, y_0, z_0)$ and contains P, as shown in Figure 13.57.

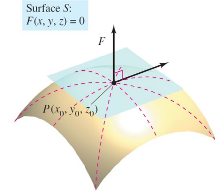


Figure 13.57

Tangent Plane and Normal Line to a Surface

DEFINITIONS OF TANGENT PLANE AND NORMAL LINE

Let F be differentiable at the point $P(x_0, y_0, z_0)$ on the surface S given by $F(x, y, z) = 0$ such that $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$.

1. The plane through P that is normal to $\nabla F(x_0, y_0, z_0)$ is called the **tangent plane to S at P.**
2. The line through P having the direction of $\nabla F(x_0, y_0, z_0)$ is called the **normal line to S at P.**

THEOREM 13.13 EQUATION OF TANGENT PLANE

If F is differentiable at (x_0, y_0, z_0), then an equation of the tangent plane to the surface given by $F(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

Example 2 – *Finding an Equation of a Tangent Plane*

Find an equation of the tangent plane to the hyperboloid given by

$$z^2 - 2x^2 - 2y^2 = 12$$

at the point (1, –1, 4).

Solution:

Begin by writing the equation of the surface as

$$z^2 - 2x^2 - 2y^2 - 12 = 0.$$

Then, considering

$$F(x, y, z) = z^2 - 2x^2 - 2y^2 - 12$$

you have

$$F_x(x, y, z) = -4x, \quad F_y(x, y, z) = -4y \quad \text{and} \quad F_z(x, y, z) = 2z.$$

Example 2 – *Solution*

cont'd

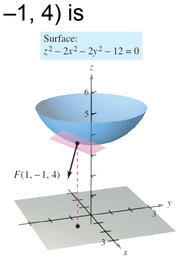
At the point (1, –1, 4) the partial derivatives are

$F_x(1, -1, 4) = -4, \quad F_y(1, -1, 4) = 4, \quad \text{and} \quad F_z(1, -1, 4) = 8.$

So, an equation of the tangent plane at (1, –1, 4) is

$$-4(x - 1) + 4(y + 1) + 8(z - 4) = 0$$
$$-4x + 4 + 4y + 4 + 8z - 32 = 0$$
$$-4x + 4y + 8z - 24 = 0$$
$$x - y - 2z + 6 = 0.$$

Figure 13.58 shows a portion of the hyperboloid and tangent plane.



Tangent plane to surface
Figure 13.58

Tangent Plane and Normal Line to a Surface

To find the equation of the tangent plane at a point on a surface given by $z = f(x, y)$, you can define the function F by

$$F(x, y, z) = f(x, y) - z.$$

Then S is given by the level surface $F(x, y, z) = 0$, and by Theorem 13.13 an equation of the tangent plane to S at the point (x_0, y_0, z_0) is

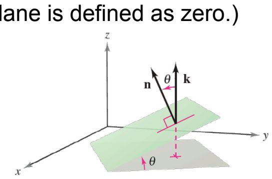
$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

The Angle of Inclination of a Plane

The Angle of Inclination of a Plane

The gradient $\nabla F(x, y, z)$ is used to determine the angle of inclination of the tangent plane to a surface.

The **angle of inclination** of a plane is defined as the angle θ ($0 \leq \theta \leq \pi/2$) between the given plane and the xy-plane, as shown in Figure 13.62. (The angle of inclination of a horizontal plane is defined as zero.)



The angle of inclination
Figure 13.62

The Angle of Inclination of a Plane

Because the vector **k** is normal to the *xy*-plane, you can use the formula for the cosine of the angle between two planes to conclude that the angle of inclination of a plane with normal vector **n** is given by

$$\cos\theta = \frac{|\mathbf{n}\cdot\mathbf{k}|}{\|\mathbf{n}\|\,\|\mathbf{k}\|} = \frac{|\mathbf{n}\cdot\mathbf{k}|}{\|\mathbf{n}\|}.$$

Angle of inclination of a plane

Example 6 – *Finding the Angle of Inclination of a Tangent Plane*

Find the angle of inclination of the tangent plane to the ellipsoid given by

$$\frac{x^2}{12} + \frac{y^2}{12} + \frac{z^2}{3} = 1$$

at the point (2, 2, 1).

Solution:

If you let

$$F(x, y, z) = \frac{x^2}{12} + \frac{y^2}{12} + \frac{z^2}{3} - 1$$

the gradient of F at the point (2, 2, 1) is given by

$$\nabla F(x, y, z) = \frac{x}{6}\mathbf{i} + \frac{y}{6}\mathbf{j} + \frac{2z}{3}\mathbf{k}$$

Example 6 – *Solution*

cont'd

$$\nabla F(2, 2, 1) = \frac{1}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}.$$

Because $\nabla F(2, 2, 1)$ is normal to the tangent plane and **k** is normal to the *xy*-plane, it follows that the angle of inclination of the tangent plane is given by

$$\cos\theta = \frac{|\nabla F(2, 2, 1)\cdot\mathbf{k}|}{\|\nabla F(2, 2, 1)\|} = \frac{2/3}{\sqrt{(1/3)^2 + (1/3)^2 + (2/3)^2}} = \sqrt{\frac{2}{3}}$$

Example 6 – *Solution*

cont'd

which implies that

$$\theta = \arccos\sqrt{\frac{2}{3}} \approx 35.3°,$$

as shown in Figure 13.63.

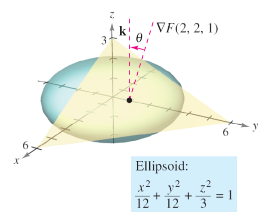


Figure 13.63

A Comparison of the Gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$

A Comparison of the Gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$

This section concludes with a comparison of the gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$.

You know that the gradient of a function f of two variables is normal to the level curves of f.

Specifically, Theorem 13.12 states that if f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0).

Having developed normal lines to surfaces, you can now extend this result to a function of three variables.

A Comparison of the Gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$

THEOREM 13.14 GRADIENT IS NORMAL TO LEVEL SURFACES

If F is differentiable at (x_0, y_0, z_0) and $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, then $\nabla F(x_0, y_0, z_0)$ is normal to the level surface through (x_0, y_0, z_0).

13.8 Extrema of Functions of Two Variables

Objectives

- Find absolute and relative extrema of a function of two variables.
- Use the Second Partials Test to find relative extrema of a function of two variables.

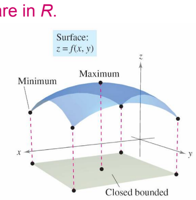
Absolute Extrema and Relative Extrema

Absolute Extrema and Relative Extrema

Consider the continuous function *f* of two variables, defined on a closed bounded region *R*. The values $f(a, b)$ and $f(c, d)$ such that

$f(a, b) \leq f(x, y) \leq f(c, d)$ (a, b) and (c, d) are in R.

for all (x, y) in *R* are called the **minimum** and **maximum** of *f* in the region *R*, as shown in Figure 13.64.



R contains point(s) at which $f(x, y)$ is a minimum and point(s) at which $f(x, y)$ is a maximum.

Figure 13.64

Absolute Extrema and Relative Extrema

A region in the plane is *closed* if it contains all of its boundary points.

The Extreme Value Theorem deals with a region in the plane that is both closed and *bounded*.

A region in the plane is called **bounded** if it is a subregion of a closed disk in the plane.

Absolute Extrema and Relative Extrema

THEOREM 13.15 EXTREME VALUE THEOREM

Let f be a continuous function of two variables x and y defined on a closed bounded region R in the xy-plane.

1. There is at least one point in R at which f takes on a minimum value.
2. There is at least one point in R at which f takes on a maximum value.

Absolute Extrema and Relative Extrema

A minimum is also called an **absolute minimum** and a maximum is also called an **absolute maximum.** As in single-variable calculus, there is a distinction made between absolute extrema and **relative extrema.**

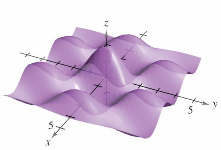
DEFINITION OF RELATIVE EXTREMA

Let f be a function defined on a region R containing (x_0, y_0).

1. The function f has a **relative minimum** at (x_0, y_0) if
$$f(x, y) \geq f(x_0, y_0)$$
for all (x, y) in an *open* disk containing (x_0, y_0).
2. The function f has a **relative maximum** at (x_0, y_0) if
$$f(x, y) \leq f(x_0, y_0)$$
for all (x, y) in an *open* disk containing (x_0, y_0).

Absolute Extrema and Relative Extrema

To say that f has a relative maximum at (x_0, y_0) means that the point (x_0, y_0, z_0) is at least as high as all nearby points on the graph of $z = f(x, y)$. Similarly, f has a relative minimum at (x_0, y_0) if (x_0, y_0, z_0) is at least as low as all nearby points on the graph. (See Figure 13.65.)



Relative extrema

Figure 13.65

Absolute Extrema and Relative Extrema

To locate relative extrema of f, you can investigate the points at which the gradient of f is **0** or the points at which one of the partial derivatives does not exist. Such points are called **critical points** of *f.*

DEFINITION OF CRITICAL POINT

Let f be defined on an open region R containing (x_0, y_0). The point (x_0, y_0) is a **critical point** of f if one of the following is true.

1. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$
2. $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Absolute Extrema and Relative Extrema

If f is differentiable and

$$\begin{aligned}\nabla f(x_0, y_0) &= f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j} \\ &= 0\mathbf{i} + 0\mathbf{j}\end{aligned}$$

then every directional derivative at (x_0, y_0) must be 0. This implies that the function has a horizontal tangent plane at the point (x_0, y_0), as shown in Figure 13.66.

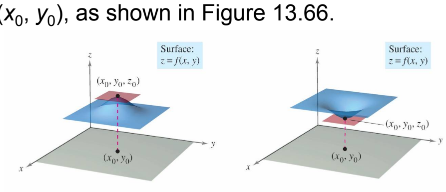


Figure 13.66

Absolute Extrema and Relative Extrema

It appears that such a point is a likely location of a relative extremum.

This is confirmed by Theorem 13.16.

THEOREM 13.16 RELATIVE EXTREMA OCCUR ONLY AT CRITICAL POINTS

If f has a relative extremum at (x_0, y_0) on an open region R, then (x_0, y_0) is a critical point of f.

Example 1 – *Finding a Relative Extremum*

Determine the relative extrema of

$f(x, y) = 2x^2 + y^2 + 8x - 6y + 20.$

Solution:

Begin by finding the critical points of *f*.

Because

$f_x(x, y) = 4x + 8$ Partial with respect to *x*

and

$f_y(x, y) = 2y - 6$ Partial with respect to *y*

are defined for all *x* and *y*, the only critical points are those for which both first partial derivatives are 0.

Example 1 – *Solution*

cont'd

To locate these points, set $f_x(x, y)$ and $f_y(x, y)$ equal to 0, and solve the equations

$4x + 8 = 0$ and $2y - 6 = 0$

to obtain the critical point (–2, 3).

By completing the square, you can conclude that for all $(x, y) \neq (-2, 3)$

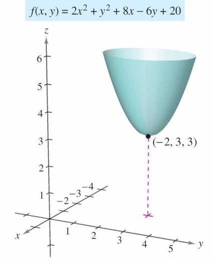
$f(x, y) = 2(x + 2)^2 + (y - 3)^2 + 3 > 3.$

So, a relative *minimum* of *f* occurs at (–2, 3).

Example 1 – *Solution*

cont'd

The value of the relative minimum is $f(-2, 3) = 3$, as shown in Figure 13.67.



The function $z = f(x, y)$ has a relative minimum at $(-2, 3)$.

Figure 13.67

The Second Partials Test

The Second Partials Test

To find relative extrema you need only examine values of *f* (*x*, *y*) at critical points. However, as is true for a function of one variable, the critical points of a function of two variables do not always yield relative maxima or minima.

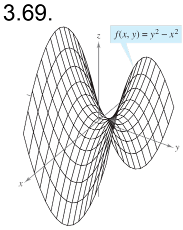
Some critical points yield **saddle points,** which are neither relative maxima nor relative minima.

The Second Partials Test

As an example of a critical point that does not yield a relative extremum, consider the surface given by

$f(x, y) = y^2 - x^2$ Hyperbolic paraboloid

as shown in Figure 13.69.



Saddle point at (0, 0, 0):
$f_x(0, 0) = f_y(0, 0) = 0$

Figure 13.69

The Second Partials Test

At the point (0, 0), both partial derivatives are 0.

The function f does not, however, have a relative extremum at this point because in any open disk centered at (0, 0) the function takes on both negative values (along the x-axis) *and* positive values (along the y-axis).

So, the point (0, 0, 0) is a saddle point of the surface.

The Second Partials Test

THEOREM 13.17 SECOND PARTIALS TEST

Let f have continuous second partial derivatives on an open region containing a point (a, b) for which

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0.$$

To test for relative extrema of f, consider the quantity

$$d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

1. If $d > 0$ and $f_{xx}(a, b) > 0$, then f has a **relative minimum** at (a, b).
2. If $d > 0$ and $f_{xx}(a, b) < 0$, then f has a **relative maximum** at (a, b).
3. If $d < 0$, then $(a, b, f(a, b))$ is a **saddle point.**
4. The test is inconclusive if $d = 0$.

Example 3 – *Using the Second Partials Test*

Find the relative extrema of

$$f(x, y) = -x^3 + 4xy - 2y^2 + 1.$$

Solution:

Begin by finding the critical points of f.

Because

$$f_x(x, y) = -3x^2 + 4y$$

and

$$f_y(x, y) = 4x - 4y$$

exist for all x and y, the only critical points are those for which both first partial derivatives are 0.

Example 3 – *Solution*

cont'd

To locate these points, set $f_x(x, y)$ and $f_y(x, y)$ equal to 0 to obtain

$$-3x^2 + 4y = 0 \text{ and } 4x - 4y = 0.$$

From the second equation you know that $x = y$, and, by substitution into the first equation, you obtain two solutions: $y = x = 0$ and $y = x = \frac{4}{3}$.

Example 3 – *Solution*

cont'd

Because

$$f_{xx}(x, y) = -6x, \quad f_{yy}(x, y) = -4, \quad \text{and} \quad f_{xy}(x, y) = 4$$

it follows that, for the critical point (0, 0),

$$d = f_{xx}(0, 0)f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = 0 - 16 < 0$$

and, by the Second Partials Test, you can conclude that (0, 0, 1) is a saddle point of f.

Example 3 – *Solution*

cont'd

Furthermore, for the critical point $\left(\frac{4}{3}, \frac{4}{3}\right)$,

$$\begin{aligned} d &= f_{xx}\left(\tfrac{4}{3}, \tfrac{4}{3}\right) f_{yy}\left(\tfrac{4}{3}, \tfrac{4}{3}\right) - \left[f_{xy}\left(\tfrac{4}{3}, \tfrac{4}{3}\right)\right]^2 \\ &= -8(-4) - 16 \\ &= 16 \\ &> 0 \end{aligned}$$

and because $f_{xx}\left(\frac{4}{3}, \frac{4}{3}\right) = -8 < 0$ you can conclude that f has a relative maximum at $\left(\frac{4}{3}, \frac{4}{3}\right)$, as shown in Figure 13.70.

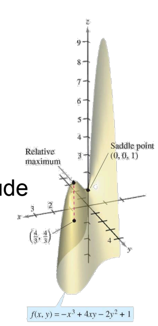


Figure 13.70

The Second Partials Test

Absolute extrema of a function can occur in two ways.

First, some relative extrema also happen to be absolute extrema.

For instance, in Example 1, $f(-2, 3)$ is an absolute minimum of the function. (On the other hand, the relative maximum found in Example 3 is not an absolute maximum of the function.)

Second, absolute extrema can occur at a boundary point of the domain.

13.9 Applications of Extrema of Functions of Two Variables

Objectives

- Solve optimization problems involving functions of several variables.
- Use the method of least squares.

Applied Optimization Problems

Example 1 – *Finding Maximum Volume*

A rectangular box is resting on the xy-plane with one vertex at the origin. The opposite vertex lies in the plane

$6x + 4y + 3z = 24$

as shown in Figure 13.73. Find the maximum volume of such a box.

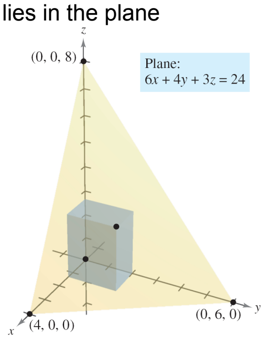


Figure 13.73

Example 1 – *Solution*

Let x, y, and z represent the length, width, and height of the box.

Because one vertex of the box lies in the plane $6x + 4y + 3z = 24$, you know that $z = \frac{1}{3}(24 - 6x - 4y)$, and you can write the volume xyz of the box as a function of two variables.

$$V(x, y) = (x)(y)\left[\tfrac{1}{3}(24 - 6x - 4y)\right]$$

$$= \tfrac{1}{3}(24xy - 6x^2y - 4xy^2)$$

Example 1 – *Solution*

cont'd

By setting the first partial derivatives equal to 0

$$V_x(x, y) = \frac{1}{3}(24y - 12xy - 4y^2) = \frac{y}{3}(24 - 12x - 4y) = 0$$

$$V_y(x, y) = \frac{1}{3}(24x - 6x^2 - 8xy) = \frac{x}{3}(24 - 6x - 8y) = 0$$

you obtain the critical points (0, 0) and $\left(\frac{4}{3}, 2\right)$.

At (0, 0) the volume is 0, so that point does not yield a maximum volume.

Example 1 – *Solution*

cont'd

At the point $\left(\frac{4}{3}, 2\right)$, you can apply the Second Partials Test.

$$V_{xx}(x, y) = -4y$$

$$V_{yy}(x, y) = \frac{-8x}{3}$$

$$V_{xy}(x, y) = \frac{1}{3}(24 - 12x - 8y)$$

Because

$$V_{xx}\left(\tfrac{4}{3}, 2\right)V_{yy}\left(\tfrac{4}{3}, 2\right) - \left[V_{xy}\left(\tfrac{4}{3}, 2\right)\right]^2 = (-8)\left(-\tfrac{32}{9}\right) - \left(-\tfrac{8}{3}\right)^2 = \tfrac{64}{3} > 0$$

and

$$V_{xx}\left(\tfrac{4}{3}, 2\right) = -8 < 0$$

Example 1 – *Solution*

cont'd

you can conclude from the Second Partials Test that the maximum volume is

$$V\left(\tfrac{4}{3}, 2\right) = \tfrac{1}{3}\left[24\left(\tfrac{4}{3}\right)(2) - 6\left(\tfrac{4}{3}\right)^2(2) - 4\left(\tfrac{4}{3}\right)(2^2)\right]$$

$$= \tfrac{64}{9} \text{ cubic units.}$$

Note that the volume is 0 at the boundary points of the triangular domain of V.

The Method of Least Squares

The Method of Least Squares

Many examples involves **mathematical models.**
For example, a quadratic model for profit.

There are several ways to develop such models; one is called the **method of least squares.**

In constructing a model to represent a particular phenomenon, the goals are simplicity and accuracy.

Of course, these goals often conflict.

The Method of Least Squares

For instance, a simple linear model for the points in Figure 13.74 is

$y = 1.8566x - 5.0246$.

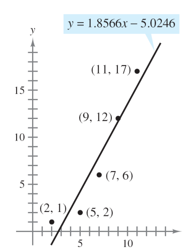


Figure 13.74

The Method of Least Squares

However, Figure 13.75 shows that by choosing the slightly more complicated quadratic model

$$y = 0.1996x^2 - 0.7281x + 1.3749$$

you can achieve greater accuracv.

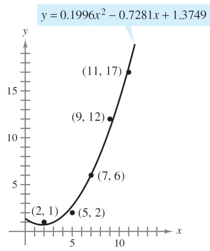


Figure 13.75

The Method of Least Squares

As a measure of how well the model $y = f(x)$ fits the collection of points

$$\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)\}$$

you can add the squares of the differences between the actual y-values and the values given by the model to obtain the **sum of the squared errors**

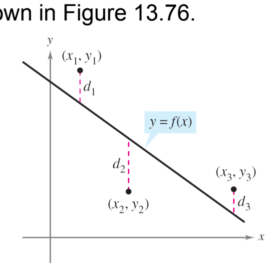
$$S = \sum_{i=1}^{n} [f(x_i) - y_i]^2.$$

Sum of the squared errors

The Method of Least Squares

Graphically, S can be interpreted as the sum of the squares of the vertical distances between the graph of f and the given points in the plane, as shown in Figure 13.76.

If the model is perfect, then $S = 0$. However, when perfection is not feasible, you can settle for a model that minimizes S.



Sum of the squared errors:
$S = d_1^2 + d_2^2 + d_3^2$

Figure 13.76

The Method of Least Squares

For instance, the sum of the squared errors for the linear model in Figure 13.74 is $S \approx 17$.

Statisticians call the *linear model* that minimizes S the **least squares regression line.**

The proof that this line actually minimizes S involves the minimizing of a function of two variables.

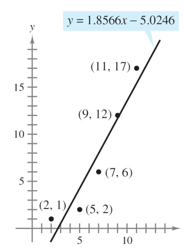


Figure 13.74

The Method of Least Squares

THEOREM 13.18 LEAST SQUARES REGRESSION LINE

The **least squares regression line** for $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ is given by $f(x) = ax + b$, where

$$a = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \quad \text{and} \quad b = \frac{1}{n}\left(\sum_{i=1}^{n} y_i - a\sum_{i=1}^{n} x_i\right).$$

The Method of Least Squares

If the x-values are symmetrically spaced about the y-axis, then $\Sigma\, x_i = 0$ and the formulas for a and b simplify to

$$a = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2} \quad \text{and} \quad b = \frac{1}{n}\sum_{i=1}^{n} y_i.$$

This simplification is often possible with a translation of the x-values.

For instance, if the x-values in a data collection consist of the years 2005, 2006, 2007, 2008, and 2009, you could let 2007 be represented by 0.

Example 3 – *Finding the Least Squares Regression Line*

Find the least squares regression line for the points (–3, 0), (–1, 1), (0, 2), and (2, 3).

Solution:

The table shows the calculations involved in finding the least squares regression line using n = 4.

x	y	xy	x^2
-3	0	0	9
-1	1	-1	1
0	2	0	0
2	3	6	4
$\sum_{i=1}^{n} x_i = -2$	$\sum_{i=1}^{n} y_i = 6$	$\sum_{i=1}^{n} x_i y_i = 5$	$\sum_{i=1}^{n} x_i^2 = 14$

Example 3 – *Solution*

cont'd

Applying Theorem 13.18 produces

$$a = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} = \frac{4(5) - (-2)(6)}{4(14) - (-2)^2} = \frac{8}{13}$$

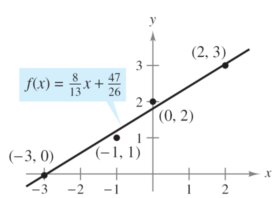
and

$$b = \frac{1}{n}\left(\sum_{i=1}^{n} y_i - a\sum_{i=1}^{n} x_i\right) = \frac{1}{4}\left[6 - \frac{8}{13}(-2)\right] = \frac{47}{26}.$$

Example 3 – *Solution*

cont'd

The least squares regression line is $f(x) = \frac{8}{13}x + \frac{47}{26}$, as shown in Figure 13.77.



Least squares regression line

Figure 13.77

13.10 Lagrange Multipliers

Objectives

- Understand the Method of Lagrange Multipliers.
- Use Lagrange multipliers to solve constrained optimization problems.
- Use the Method of Lagrange Multipliers with two constraints.

Lagrange Multipliers

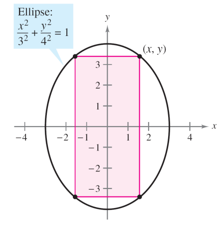
Lagrange Multipliers

Many optimization problems have restrictions, or **constraints,** on the values that can be used to produce the optimal solution. Such constraints tend to complicate optimization problems because the optimal solution can occur at a boundary point of the domain. In this section, you will study an ingenious technique for solving such problems. It is called the **Method of Lagrange Multipliers.**

To see how this technique works, suppose you want to find the rectangle of maximum area that can be inscribed in the ellipse given by $\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1.$

Lagrange Multipliers

Let (x, y) be the vertex of the rectangle in the first quadrant, as shown in Figure 13.78.



Objective function: $f(x, y) = 4xy$

Figure 13.78

Lagrange Multipliers

Because the rectangle has sides of lengths $2x$ and $2y$, its area is given by

$f(x, y) = 4xy.$ Objective function

You want to find x and y such that $f(x, y)$ is a maximum.

Your choice of (x, y) is restricted to first-quadrant points that lie on the ellipse

$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1.$ Constraint

Lagrange Multipliers

Now, consider the constraint equation to be a fixed level curve of

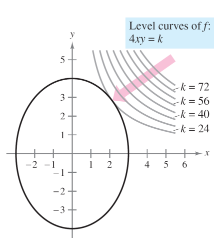
$$g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2}.$$

The level curves of f represent a family of hyperbolas $f(x, y) = 4xy = k$. In this family, the level curves that meet the given constraint correspond to the hyperbolas that intersect the ellipse.

Moreover, to maximize $f(x, y)$, you want to find the hyperbola that just barely satisfies the constraint.

Lagrange Multipliers

The level curve that does this is the one that is *tangent* to the ellipse, as shown in Figure 13.79.



Constraint: $g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$

Figure 13.79

Lagrange Multipliers

If $\nabla f(x, y) = \lambda \nabla g(x, y)$ then scalar λ is called **Lagrange multiplier.**

THEOREM 13.19 LAGRANGE'S THEOREM

Let f and g have continuous first partial derivatives such that f has an extremum at a point (x_0, y_0) on the smooth constraint curve $g(x, y) = c$. If $\nabla g(x_0, y_0) \neq \mathbf{0}$, then there is a real number λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

Lagrange Multipliers

METHOD OF LAGRANGE MULTIPLIERS

Let f and g satisfy the hypothesis of Lagrange's Theorem, and let f have a minimum or maximum subject to the constraint $g(x, y) = c$. To find the minimum or maximum of f, use the following steps.

1. Simultaneously solve the equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = c$ by solving the following system of equations.

$$f_x(x, y) = \lambda g_x(x, y)$$
$$f_y(x, y) = \lambda g_y(x, y)$$
$$g(x, y) = c$$

2. Evaluate f at each solution point obtained in the first step. The largest value yields the maximum of f subject to the constraint $g(x, y) = c$, and the smallest value yields the minimum of f subject to the constraint $g(x, y) = c$.

Constrained Optimization Problems

Example 1 – *Using a Lagrange Multiplier with One Constraint*

Find the maximum value of $f(x, y) = 4xy$ where $x > 0$ and $y > 0$, subject to the constraint $(x^2/3^2) + (y^2/4^2) = 1$.

Solution:

To begin, let

$$g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1.$$

By equating $\nabla f(x, y) = 4y\mathbf{i} + 4x\mathbf{j}$ and $\lambda \nabla g(x, y) = (2\lambda x/9)\mathbf{i} + (\lambda y/8)\mathbf{j}$, you can obtain the following system of equations.

$$4y = \frac{2}{9}\lambda x \qquad f_x(x, y) = \lambda g_x(x, y)$$

$$4x = \frac{1}{8}\lambda y \qquad f_y(x, y) = \lambda g_y(x, y)$$

Example 1 – *Solution*

cont'd

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1 \qquad \text{Constraint}$$

From the first equation, you obtain $\lambda = 18y/x$, and substitution into the second equation produces

$$4x = \frac{1}{8}\left(\frac{18y}{x}\right)y \quad \Rightarrow \quad x^2 = \frac{9}{16}y^2.$$

Substituting this value for x^2 into the third equation produces

$$\frac{1}{9}\left(\frac{9}{16}y^2\right) + \frac{1}{16}y^2 = 1 \quad \Rightarrow \quad y^2 = 8.$$

Example 1 – *Solution*

cont'd

So, $y = \pm 2\sqrt{2}$.

Because it is required that $y > 0$, choose the positive value and find that

$$x^2 = \frac{9}{16}y^2$$
$$= \frac{9}{16}(8) = \frac{9}{2}$$
$$x = \frac{3}{\sqrt{2}}.$$

So, the maximum value of f is

$$f\left(\frac{3}{\sqrt{2}}, 2\sqrt{2}\right) = 4xy = 4\left(\frac{3}{\sqrt{2}}\right)\left(2\sqrt{2}\right) = 24.$$

Constrained Optimization Problems

Economists call the Lagrange multiplier obtained in a production function the **marginal productivity of money.** For instance, the marginal productivity of money is λ and x represents the units of labor and y represents the units of capital then the marginal productivity of money at $x = 250$ and $y = 50$ is

$$\lambda = \frac{x^{-1/4}y^{1/4}}{2} = \frac{(250)^{-1/4}(50)^{1/4}}{2} \approx 0.334$$

which means that for each additional dollar spent on production, an additional 0.334 unit of the product can be produced.

The Method of Lagrange Multipliers with Two Constraints

The Method of Lagrange Multipliers with Two Constraints

For optimization problems involving *two* constraint functions *g* and *h*, you can introduce a second Lagrange multiplier, μ (the lowercase Greek letter mu), and then solve the equation

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

where the gradient vectors are not parallel, as illustrated in Example 5.

Example 5 – *Optimization with Two Constraints*

Let $T(x, y, z) = 20 + 2x + 2y + z^2$ represent the temperature at each point on the sphere $x^2 + y^2 + z^2 = 11$.
Find the extreme temperatures on the curve formed by the intersection of the plane $x + y + z = 3$ and the sphere.

Solution:

The two constraints are

$$g(x, y, z) = x^2 + y^2 + z^2 = 11$$

and

$$h(x, y, z) = x + y + z = 3.$$

Example 5 – *Solution*

cont'd

Using
$$\nabla T(x, y, z) = 2\mathbf{i} + 2\mathbf{j} + 2z\mathbf{k}$$
$$\lambda \nabla g(x, y, z) = 2\lambda x\mathbf{i} + 2\lambda y\mathbf{j} + 2\lambda z\mathbf{k}$$
and
$$\mu \nabla h(x, y, z) = \mu\mathbf{i} + \mu\mathbf{j} + \mu\mathbf{k}$$

you can write the following system of equations.

$$2 = 2\lambda x + \mu \qquad T_x(x, y, z) = \lambda g_x(x, y, z) + \mu h_x(x, y, z)$$
$$2 = 2\lambda y + \mu \qquad T_y(x, y, z) = \lambda g_y(x, y, z) + \mu h_y(x, y, z)$$
$$2z = 2\lambda z + \mu \qquad T_z(x, y, z) = \lambda g_z(x, y, z) + \mu h_z(x, y, z)$$
$$x^2 + y^2 + z^2 = 11 \qquad \text{Constraint 1}$$
$$x + y + z = 3 \qquad \text{Constraint 2}$$

Example 5 – *Solution*

cont'd

By subtracting the second equation from the first, you can obtain the following system.

$$\lambda(x - y) = 0$$
$$2z(1 - \lambda) - \mu = 0$$
$$x^2 + y^2 + z^2 = 11$$
$$x + y + z = 3$$

From the first equation, you can conclude that $\lambda = 0$ or $x = y$.

If $\lambda = 0$, you can show that the critical points are (3, –1, 1) and (–1, 3, 1).

Example 5 – *Solution*

cont'd

If $\lambda \neq 0$, then $x = y$ and you can show that the critical points occur when $x = y = \left(3 \pm 2\sqrt{3}\right)/3 \quad \left(3 \mp 4\sqrt{3}\right)/3.$

Finally, to find the optimal solutions, compare the temperatures at the four critical points.

$$T(3, -1, 1) = T(-1, 3, 1) = 25$$
$$T\left(\frac{3 - 2\sqrt{3}}{3}, \frac{3 - 2\sqrt{3}}{3}, \frac{3 + 4\sqrt{3}}{3}\right) = \frac{91}{3} \approx 30.33$$
$$T\left(\frac{3 + 2\sqrt{3}}{3}, \frac{3 + 2\sqrt{3}}{3}, \frac{3 - 4\sqrt{3}}{3}\right) = \frac{91}{3} \approx 30.33$$

So, $T = 25$ is the minimum temperature and $T = \frac{91}{3}$ is the maximum temperature on the curve.