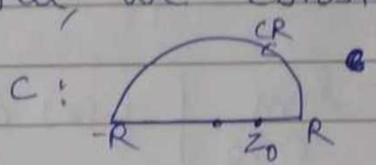
Indented contour:-

Evaluation of improper integral when simple pole lies of real axis:-

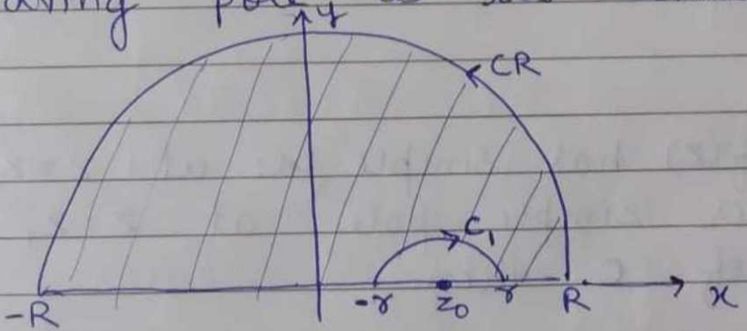
Let $I = \int_{-\infty}^{\infty} f(x)\,dx$.

To evaluate the integral, we consider

$\oint_C f(z)dz$.



If $f(z)$ has pole on real axis then ~~thees~~ ~~thees poles~~ cauchy ~~integral~~ residue theorem is not applicable for the contour C as $f(z)$ is not analytic on closed curve C.
~~ie~~. In this case pole is deleted by indenting the contour. which is done by drawing a small circle of radius r, having pole as its center.



$C_1: |z-z_0|=r$

Now cauchy's Residue theorem is applicable for defined $(C-C_1)$ region

Theorem :- If $f(z)$ has simple pole at $z=z_0$ on real axis then

$$\lim_{r\to 0} \int_{C_1} f(z)\,dz = \pi i \times \operatorname{Res}[f(z); z=z_0]$$

or

$\lim_{r\to 0} \int_{C_1} f(z)\,dz$ can be computed by

putting $z = re^{i\theta}$, $dz = ri\,e^{i\theta}\,d\theta$

i.e. $\lim_{r\to 0} \int_{0}^{\pi} f(re^{i\theta})\, ri\, e^{i\theta}\,d\theta$, can be computed.

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If $f(z)$ has simple pole at $z=z_0$ and another simple pole at $z=z_1$ in upper half of C then

$$\int_{-\infty}^{\infty} f(x)\,dx = \pi i \times \operatorname{Res}(f(z); z=z_0) + 2\pi i \times \operatorname{Res}(f(z); z=z_1)$$

1) Evaluate $\int_0^{\infty} \frac{\sin mx}{x}\, dx$

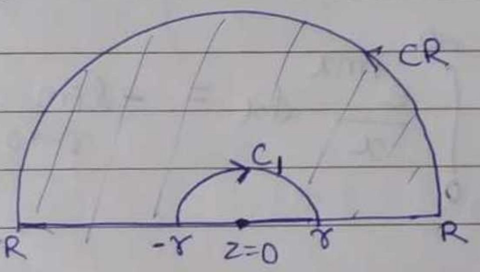
Ans consider $\int_C \frac{\sin mz}{z}\, dz$

$=$ ~~real~~ imaginary part of $\int_C \frac{e^{imz}}{z}\, dz$

$f(z) = \frac{e^{imz}}{z}$

pole of $f(z)$:- $z=0$, which lies on real axis, therefor, we consider indented contour. C as

C :



where $C_1 : |z-0| = r$

since there is no other pole within C i.e by cauchy's integral theorem

$$\oint_C f(z)\, dz = 0$$

C consist of
(1) real axis from $-R$ to $-r$
(2) along C_1, s.t θ varies from π to 0
(3) real axis from r to R
(4) along C_R s.t $0 \le \theta \le \pi$

$$\int_{-R}^{-r} \frac{e^{imx}}{x}\,dx + \int_{C_1 \uparrow} \frac{e^{imz}}{z}\,dz + \int_{r}^{R} \frac{e^{imx}}{x}\,dx + \int_{C_R} \frac{e^{imz}}{z}\,dz = 0$$

as $R \to \infty$, $r \to 0$ we have

$$\int_{-\infty}^{0} \frac{e^{imx}}{x}\,dx + \lim_{r \to 0} \int_{C_1 \uparrow} \frac{e^{imz}}{z}\,dz + \int_{0}^{\infty} \frac{e^{imx}}{x}\,dx + \lim_{R \to \infty} \int_{C_R} \frac{e^{imz}}{z}\,dz = 0$$

by Jordan lemma

$$\lim_{R \to \infty} \int_{C_R} \frac{e^{imz}}{z}\,dz = 0$$

$$\int_{-\infty}^{0} \frac{e^{imx}}{x}\,dx + \int_{0}^{\infty} \frac{e^{imx}}{x}\,dx = -\lim_{r \to 0} \int_{C_1 \uparrow} \frac{e^{imz}}{z}\,dz$$

or
$$\int_{-\infty}^{\infty} \frac{e^{imx}}{x}\,dx = \lim_{r \to 0} \int_{C_1} \frac{e^{imz}}{z}\,dz \quad \text{——} (1)$$

using

$$\lim_{r \to 0} \int_{C_1} f(z)\,dz = \pi i \times \operatorname{Res}(f(z); z=0)$$

we have

$$\lim_{r \to 0} \int_{C_1} \frac{e^{imz}}{z}\,dz = \pi i \times \lim_{z \to 0} e^{imz}$$
$$= \pi i$$

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$$\boxed{\int_{-\infty}^{\infty} \frac{e^{imx}}{x}\, dx = \pi i}$$

or $\lim_{r\to 0} \int_{C_1} \frac{e^{imz}}{z}\, dz$ can be evaluated as

$C_1 : |z| = r, \; z = re^{i\theta}$

$$= \lim_{r\to 0} \int_0^{\pi} \frac{e^{imre^{i\theta}}}{re^{i\theta}}\, r i e^{i\theta}\, d\theta$$

$$= \int_0^{\pi} \lim_{r\to 0} \left(\frac{e^{imre^{i\theta}}}{\cancel{r}\cancel{e^{i\theta}}}\, \cancel{r} i \cancel{e^{i\theta}} \right) d\theta$$

$$= \int_0^{\pi} i\, d\theta = i\pi$$

ie $\int_{-\infty}^{\infty} \frac{e^{imx}}{x}\, dx = \pi i$

$$\int_{-\infty}^{\infty} \left(\frac{\cos mx + i \sin mx}{x} \right) dx = \pi i$$

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x}\, dx = 0 \quad , \quad \int_{-\infty}^{\infty} \frac{\sin mx}{x} = \pi$$

Obviously as its odd function

$$2\int_0^\infty \frac{\sin mx}{x}\,dx = \pi$$

$$\boxed{\int_0^\infty \frac{\sin mx}{x}\,dx = \frac{\pi}{2}}$$

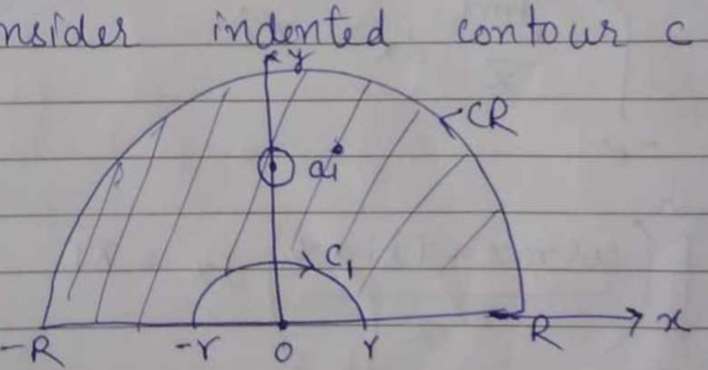
(2) :- solve $\displaystyle\int_{-\infty}^{\infty} \frac{\sin mx}{x(x^2+a^2)}\,dx$

Sol^n consider $f(z) = \dfrac{e^{imz}}{z(z^2+a^2)}$

poles are

$$z=0,\ z=\pm ai$$

$z=0$, lies on the real axis, therefor we consider indented contour c



$z=-ai$, lies outside the contour

and $z = ai$, lies inside, therefor by cauchy Residue theorem

$$\oint_C f(z) = 2\pi i \times \text{Residue of } f(z) \text{ at } z = ai$$

$$\text{Res at } (z = ai) = \lim_{z \to ai} \cancel{(z-ai)} \frac{e^{imz}}{z(z+ai)\cancel{(z-ai)}}$$

$$= \frac{e^{-am}}{(2ai)ai} = \frac{e^{-am}}{-2a^2}$$

ie $$\oint_C f(z)\, dz = 2\pi i \times \frac{e^{-am}}{-2a^2}$$

$$\left[\oint_C f(z)\, dz = -\frac{\pi i e^{-am}}{a^2} \right]$$

Now

$$\int_{-R}^{-r} \frac{e^{imx}}{x(x^2+a^2)} dx + \int_{C_1} \frac{e^{imz}}{z(z^2+a^2)} dz + \int_{r}^{R} \frac{e^{imx}}{x(x^2+a^2)} + \int_{CR} \frac{e^{imz}}{z(z^2+a^2)}$$

$$= -\frac{\pi i e^{-am}}{a^2}$$

as $R \to \infty$, $r \to 0$

we have

$$\int_{-\infty}^{0} \frac{e^{imx}}{x(x^2+a^2)} dx + \int_{0}^{\infty} \frac{e^{imx}}{x(x^2+a^2)} dx + \int_{C_1} \lim_{r \to 0} \frac{e^{imz}}{z(z^2+a^2)} dz$$

$$+ \lim_{R \to \infty} \int_{CR} \frac{e^{imz}}{z(z^2+a^2)} dz = -\frac{\pi i e^{-am}}{a^2}$$

classmate
Date
Page

by Jordan's lemma $\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$

ie we have

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{x(x^2+a^2)}\,dx = -\lim_{r\to 0} \int_{C_1 \curvearrowright} \frac{e^{imz}}{z(z^2+a^2)}\,dx - \frac{\pi i e^{-am}}{a^2}$$

$$= \lim_{r\to 0} \int_{C_1} \frac{e^{imz}}{z(z^2+a^2)}\,dx - \frac{\pi i e^{am}}{a^2}$$

$$= \pi i \times \operatorname{Res}(f(z); z=0) - \frac{\pi i e^{-am}}{a^2}$$

$$= \pi i \times \lim_{z\to 0} \left(\frac{e^{imz}}{z^2+a^2}\right) - \frac{\pi i e^{-am}}{a^2}$$

$$= \frac{\pi i}{a^2} - \frac{\pi i e^{-am}}{a^2} = \frac{\pi}{a^2}(1-e^{-am})\,i$$

comparing real and imaginary parts

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x(x^2+a^2)}\,dx = 0, \quad \int_{-\infty}^{\infty} \frac{\sin mx}{x(x^2+a^2)}\,dx = \frac{\pi}{a^2}(1-e^{-am})$$

ie $$\boxed{\int_{-\infty}^{\infty} \frac{\sin mx}{x(x^2+a^2)}\,dx = \frac{\pi}{a^2}(1-e^{-am})}$$