Module 6.2:

Fourier Sine Transform:

we have

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin(px) \left(\int_0^\infty \sin(pt)\, f(t)\, dt \right) dp$$

which is the Fourier Sine integral.

Taking $\boxed{F_s(P) = \int_0^\infty \sin(pt)\, f(t)\, dt}$, then

we get $$f(x) = \frac{2}{\pi} \int_0^\infty \sin(px)\, F_s(p)\, dp$$

The function $F_s(P)$ is the Fourier Sine transform (or, infinite Fourier sine transform) of $f(x)$ and $f(x)$ is the inverse Fourier sine transform of $F_s(P)$.

Fourier Cosine transform:

we have

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos(px) \left(\int_0^\infty \cos(pt)\, f(t)\, dt \right) dp$$

which is the Fourier Cosine integral.

Taking $\boxed{F_c(p) = \int_0^\infty \cos(pt)\, f(t)\, dt}$, then we get

$$f(x) = \frac{1}{2\pi} \int_0^\infty \cos(px)\, F_c(p)\, dp$$

The function $F_c(p)$ is the Fourier cosine transform (or, infinite Fourier cosine transform) of $f(x)$ and $f(x)$ is the inverse Fourier cosine transform of $F_c(p)$.

<u>Problems:</u>

1. Find the Fourier sine and cosine transforms of $f(x) = x$.

<u>Sol:</u> Fourie sine transform of $f(x)$ is

$$F_s(p) = \int_0^\infty \sin(px)\, f(x)\, dx$$

$$= \int_0^\infty x \sin px \, dx$$

and Fourier cosine transform of $f(x)$ is

$$F_c(p) = \int_0^\infty \cos(px)\, f(x)\, dx = \int_0^\infty x \cos(px)\, dx$$

Now, $$\int_0^\infty x\, e^{-ipx}\, dx = \left[x \cdot \frac{e^{-ipx}}{-ip}\right]_0^\infty - \left[1 \cdot \frac{e^{-ipx}}{(-ip)^2}\right]_0^\infty$$

$$\Rightarrow \int_0^\infty x \cos(px) - i \sin(px)\, dx = 0 + \frac{1}{p^2}(0-1) \quad (\because i^2 = -1)$$

$$\Rightarrow \int_0^\infty x \cos px\, dx - i \int_0^\infty x \sin px\, dx = -\frac{1}{p^2}$$

Equating real and imaginary parts,

we get

$$\int_0^\infty x \sin px\, dx = 0 \text{ and } \int_0^\infty x \cos px\, dx = \frac{-1}{p^2}$$

Hence $F_s(P) = 0$ and $F_c(P) = \frac{-1}{p^2}$.

2. Find the Fourier sine and cosine transforms of $f(x) = \frac{e^{-ax}}{x}$ $(a>0)$.

Sol: (i) Fourier sine transform of $f(x)$ is

$$F_s\{f(x)\} = F_s(p) = \int_0^\infty f(x) \sin(px)\, dx$$

$$\Rightarrow F_s\{f(x)\} = \int_0^\infty \frac{e^{-ax}}{x} \sin(px)\, dx$$

Differentiating both sides w.r.to p, we get

$$\frac{d}{dp}(F_s(p)) = \int_0^\infty \frac{e^{-ax}}{x} x \cos(px)\, dx$$

$$= \int_0^\infty e^{-ax} \cos(px)\, dx$$

$$= \frac{a}{a^2+p^2}$$

Integrating both sides w.r.t. p, we get

$$F_s(p) = \int \frac{a}{a^2+p^2} dp \Rightarrow F(p) = \tan^{-1}\left(\frac{p}{a}\right) + c_1$$

Similarly, Fourier Cosine Transform of $f(x)$ is $F_c(p) = \frac{-1}{2} \cdot \log(p^2+a^2) + c_2$

3. Find the Fourier sine and cosine transforms of $f(x) = e^{-ax}$ $(a>0)$ and hence evaluate

(i) $\int_0^\infty \frac{\cos px}{a^2+p^2}\,dp$ and (ii) $\int_0^\infty \frac{p \sin px}{a^2+p^2}\,dp$

Sol: (i) $F_c\{f(x)\} = \int_0^\infty f(x) \cos px \, dx$

$$= \int_0^\infty e^{-ax} \cos px \, dx$$

$$= \left. \frac{e^{-ax}}{a^2+p^2} \left[-a \cos px + p \sin px\right] \right]_0^\infty$$

$$= 0 - \frac{1}{a^2+p^2}(-a \cdot 1 + p \cdot 0)$$

$$\therefore \quad F_c(p) = \frac{a}{a^2+p^2}$$

By the inverse Fourier cosine transform, we get

$$f(x) = \frac{2}{\pi} \int_0^\infty F_c(p) . \cos px \, dp$$

$$\Rightarrow e^{-ax} = \frac{2}{\pi}\int_0^\infty \frac{a}{a^2+p^2} \cos px \, dp$$

or, $$\int_0^\infty \frac{\cos px}{a^2+p^2} dp = \frac{\pi}{2a} e^{-ax}.$$

(ii) Fourier Sine transform of $f(x)$ is

$$F_s(P) = \int_0^\infty f(x) \sin(px) \, dx$$

$$= \int_0^\infty e^{-ax} \sin px \, dx$$

$$= \left[\frac{e^{-ax}}{a^2+p^2} \left[-a \sin px - p\cos px\right]\right]_0^\infty$$

$$= 0 - \frac{1}{a^2+p^2} [0 - P]$$

$$= \frac{p}{a^2+p^2}$$

By the inverse Fourier Sine transform of $F_s(P)$,

we have

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(P) \sin px \, dp$$

$$\Rightarrow e^{-ax} = \frac{2}{\pi} \int_0^\infty \frac{p}{a^2+p^2} \sin px \, dp$$

Hence $\int_0^\infty \frac{p}{a^2+p^2} \sin px \, dp = \frac{\pi}{2} e^{-ax}$

4. Find the Fourier Sine transform of $\frac{x}{a^2+x^2}$ $(a>0)$ and Fourier Cosine transform of $\frac{1}{a^2+x^2}$ $(a>0)$.

Sol: we have, $F_s\{e^{-ax}\} = \frac{p}{a^2+p^2}$

So, the inverse Fourier Sine transform of e^{-ax} is

$$e^{-ax} = \frac{2}{\pi}\int_0^\infty F_s\{e^{-ax}\} \sin px \, dp$$

$$= \frac{2}{\pi}\int_0^\infty \frac{p}{a^2+p^2} \sin px \, dp$$

or, $$\int_0^\infty \frac{p}{a^2+p^2} \sin px \, dp = \frac{\pi}{2} e^{-ax}$$

Changin x to p and p to x, we get

$$\int_0^\infty \frac{x}{a^2+x^2} \sin(xp) \, dx = \frac{\pi}{2} e^{-ap}$$

Hence $$\boxed{F_s\left\{\frac{x}{a^2+x^2}\right\} = \frac{\pi}{2} e^{-ap}}$$

Also, we have $F_c\{e^{-ax}\} = \frac{a}{a^2+p^2}$

The inverse Fourier Cosine transform of e^{-ax} is

$$e^{-ax} = \frac{2}{\pi}\int_0^\infty F_c\{e^{-ax}\} \cos px \, dp$$

$$= \frac{2}{\pi}\int_0^\infty \frac{a}{a^2+p^2} \cos px \, dp$$

or, $$\int_0^\infty \frac{a}{a^2+p^2} \cos px \, dp = \frac{\pi}{2} e^{-ax}$$

or, $$\int_0^\infty \frac{\cos px}{a^2+p^2} \, dp = \frac{\pi}{2a} e^{-ax}$$

Changing x to p and p to x, we get

$$\int_0^\infty \frac{\cos(xp)}{a^2+x^2} \, dx = \frac{\pi}{2a} e^{-ap}$$

Hence, $$\int_0^\infty \frac{1}{a^2+x^2} \cos(px) \, dx = \frac{\pi}{2a} e^{-ap}$$

i.e., $$F_c\left\{\frac{1}{a^2+x^2}\right\} = \frac{\pi}{2a} e^{-ap}.$$

5. Find the Fourier Sine transform of $e^{-|x|}$ and hence evaluate $\int_0^\infty \frac{x \sin(nx)}{1+x^2} \, dx$ $\boxed{\text{Answer} = \frac{\pi}{2} e^{-n}}$

6. Find the Fourier cosine transform of

$$f(x) = \begin{cases} x & \text{if } 0<x<1 \\ 2-x & \text{if } 1<x<2 \\ 0 & \text{if } x>2 \end{cases}$$

Answer: $F_c\{f(x)\} = \frac{2}{p^2}\cos p\,(1-\cos p)$

7. Find the Fourier Sine and Cosine transforms of $2e^{-5x} + 5e^{-2x}$

Answer: $F_s\{f(x)\} = \frac{2p}{25+p^2} + \frac{5p}{4+p^2}$

and $F_c\{f(x)\} = \frac{2(5)}{25+p^2} + \frac{5(2)}{4+p^2}$.