Consider a fifth-order lowpass Butterworth filter with a passband of 1 KHz and a maximum passband attenuation of 1 dB. What is the actual attenuation in dB, of the lowpass filter at a frequency of 2 KHz?

The magnitude frequency response of a lowpass Butterworth filter is

$$|H(j\Omega)| = \frac{1}{\left[1+\left(\frac{\Omega}{\Omega_C}\right)^{2N}\right]^{\frac{1}{2}}}$$

$$\Rightarrow \quad 20\log|H(j\Omega)| = -10\log\left[1+\left(\frac{\Omega}{\Omega_C}\right)^{2N}\right]$$

Given, $20\log|H(j\Omega)| = -1$ at $\Omega = 2\pi \times 1 \times 10^3$ rad/sec and $N = 5$.

Hence $-1 = -10\log\left[1+\left(\frac{2\pi \times 1 \times 10^3}{\Omega_C}\right)^{10}\right]$

Solving, we get $\Omega_C = 7192.21$ rad/sec.

We are now required to find $20 \log |H(j\Omega)|$ at $\Omega = 2\pi \times 2 \times 10^3 = 4\pi \times 10^3$ rad/sec.

Using equation (4.12), we get

$$20 \log |H(j\Omega)|_{\Omega=4\pi\times 10^3} = -10 \log \left[1 + \left(\frac{2\pi \times 2 \times 10^3}{7192.21} \right)^{10} \right]$$

$$= -24.25 \text{ dB}$$

Hence, the stopband attenuation $= A_S = 24.25$ dB.

Find the order N of a lowpass Butterworth filter to meet the following specifications.

$$\delta_P = 0.001, \qquad \delta_S = 0.001$$
$$\Omega_P = 1\ \text{rad/sec}, \qquad \Omega_S = 2\ \text{rad/sec}$$

Discrimination factor, $d = \sqrt{\dfrac{(1-\delta_P)^{-2}-1}{\delta_S^{-2}-1}}$

$= 4.4755 \times 10^{-5}$

Selectivity factor, $K = \dfrac{\Omega_P}{\Omega_S} = 0.5$

We know that, $N \geqslant \dfrac{\log\left(\frac{1}{d}\right)}{\log\left(\frac{1}{K}\right)}$

$\Rightarrow N \geqslant 14.45$

$\Rightarrow N = 15$

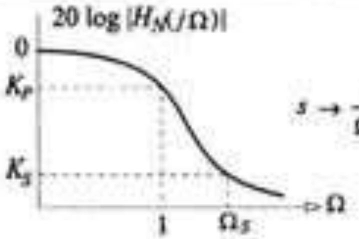
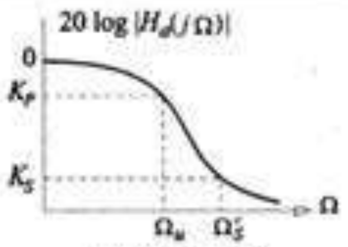
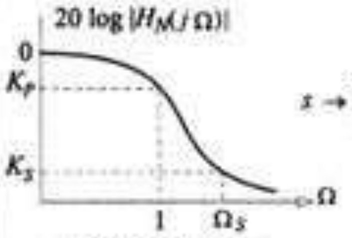
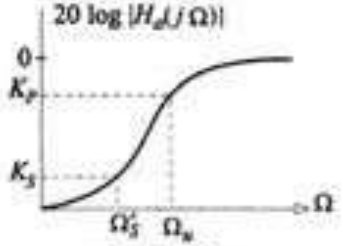
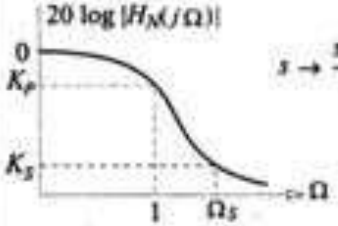
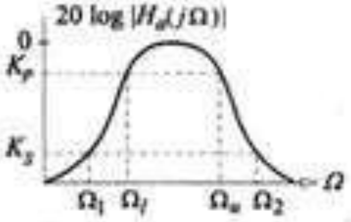
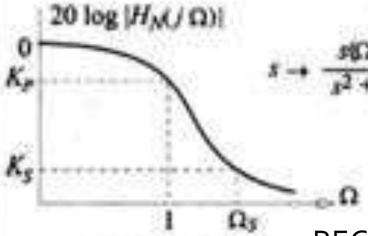
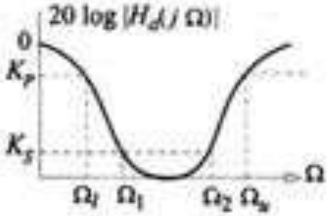
Alternate method

$$K_P = 20\log(1-\delta_P) = -8.69 \times 10^{-3}$$
$$K_S = 20\log\delta_S = -60$$
$$N = \frac{\log\left[\left(10^{\frac{-K_P}{10}}-1\right)\Big/\left(10^{\frac{-K_S}{10}}-1\right)\right]}{2\log\left(\frac{\Omega_P}{\Omega_S}\right)}$$
$$= 14.45 = 15 \quad \text{(rounded off value).}$$

Frequency transformation

Frequency transformation for analogue filters (prototype low-pass filter has band edge frequency $\Omega_c = 1$).

Type of transformation	Transformation Function
Low-pass $\rightarrow$ low-pass	$s \rightarrow \frac{s}{\Omega_c}$
Low-pass $\rightarrow$ high-pass	$s \rightarrow \frac{\Omega_c}{s}$
Low-pass $\rightarrow$ bandpass	$s \rightarrow \frac{s^2+\Omega_u\Omega_l}{s(\Omega_u-\Omega_l)}$
Low-pass $\rightarrow$ bandstop	$s \rightarrow \frac{s(\Omega_u-\Omega_l)}{s^2+\Omega_u\Omega_l}$

Prototype frequency response		Transformed frequency response	Backward design equations
 Lowpass, $H_N(s)$	$s \rightarrow \frac{s}{\Omega_u}$	 Lowpass, $H_d(s)$	$\Omega_S = \frac{\Omega'_S}{\Omega_u}$
 Lowpass, $H_N(s)$	$s \rightarrow \frac{\Omega_u}{s}$	 Highpass, $H_d(s)$	$\Omega_S = \frac{\Omega_u}{\Omega'_S}$
 Lowpass, $H_N(s)$	$s \rightarrow \frac{s^2 + \Omega_u\Omega_l}{s(\Omega_u - \Omega_l)}$	 Bandpass, $H_d(s)$	$\Omega_S = \text{Min}(\lvert A\rvert, \lvert B\rvert)$ $A = \frac{-\Omega_1^2 + \Omega_l\Omega_u}{\Omega_1(\Omega_u - \Omega_l)}$ $B = \frac{\Omega_2^2 - \Omega_l\Omega_u}{\Omega_2(\Omega_u - \Omega_l)}$
 Lowpass, $H_N(s)$	$s \rightarrow \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u\Omega_l}$	 Bandstop, $H_d(s)$	$\Omega_S = \text{Min}(\lvert A\rvert, \lvert B\rvert)$ $A = \frac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^2 + \Omega_l\Omega_u}$ $B = \frac{\Omega_2(\Omega_u - \Omega_l)}{-\Omega_2^2 + \Omega_u\Omega_l}$

Ex.1 Let $H(s) = \frac{1}{s^2+s+1}$ represent the transfer function of a lowpass filter (not Butterworth) with a passband of 1 rad/sec. Use frequency transformations to find the transfer functions of the following analog filters.

a. A lowpass filter with a passband of 10 rad/sec.

b. A highpass filter with a cutoff frequency of 1 rad/sec.

c. A highpass filter with a cutoff frequency of 10 rad/sec.

d. A bandpass filter with a passband of 10 rad/sec and a center frequency of 100 rad/sec.

e. A bandstop filter with a stopband of 2 rad/sec and a center frequency of 10 rad/sec.

☐ Solution

Given
$$H(s) = \frac{1}{s^2 + s + 1}$$

a. The lowpass-to-lowpass transformation is

$$s \longrightarrow \frac{s}{\Omega_u}$$

Hence, the required lowpass filter is

$$\begin{aligned} H_a(s) &= H(s)\Big|_{s \longrightarrow \frac{s}{10}} \\ &= \frac{1}{\left(\frac{s}{10}\right)^2 + \left(\frac{s}{10}\right) + 1} = \frac{100}{s^2 + 10s + 100} \end{aligned}$$

b. The lowpass-to-highpass transformation is

$$s \longrightarrow \frac{\Omega_u}{s}$$

Hence, the required highpass filter is

$$\begin{aligned} H_a(s) &= H(s)\Big|_{s \longrightarrow \frac{1}{s}} \\ &= \frac{s^2}{s^2+s+1} \end{aligned}$$

c.

$$\begin{aligned} H_a(s) &= \frac{1}{s^2+s+1}\Bigg|_{s \longrightarrow \frac{10}{s}} \\ &= \frac{1}{\left(\frac{10}{s}\right)^2 + \frac{10}{s} + 1} \\ &= \frac{s^2}{s^2+10s+100} \end{aligned}$$

d. The lowpass-to-bandpass transformation is

$$s \longrightarrow \frac{s^2 + \Omega_u \Omega_l}{s(\Omega_u - \Omega_l)} = \frac{s^2 + \Omega_o^2}{s B_o}$$

where $\Omega_o = \sqrt{\Omega_u \Omega_l}$ is the center frequency of the bandpass filter and $B_o = \Omega_u - \Omega_l$ is the width of the passband.

Hence, the required bandpass filter is

$$\begin{aligned} H_a(s) &= \left. \frac{1}{s^2 + s + 1} \right|_{s \longrightarrow \frac{s^2 + 10^4}{10s}} \\ &= \frac{100s^2}{s^4 + 10s^3 + 20100s^2 + 10^5 s + 10^8} \end{aligned}$$

e. The lowpass-to-bandstop transformation is

$$s \longrightarrow \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u \Omega_l} = \frac{s B_o}{s^2 + \Omega_o^2}$$

Hence, the required bandstop filter is

$$H_a(s) = \left. \frac{1}{s^2 + s + 1} \right|_{s \longrightarrow \frac{2s}{s^2+100}}$$

$$= \frac{(s^2 + 100)^2}{s^4 + 2s^3 + 204s^2 + 200s + 10^4}$$

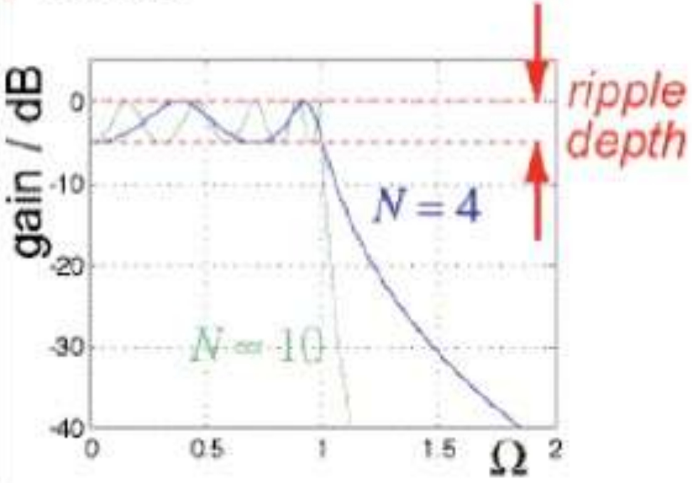
Module 3_CONTENT

- Design techniques for analog filter
  - Butterworth and Chebyshev approximations,
- Frequency transformation, Properties - Constant group delay and zero phase filters

Chebyshev I Filter

- Equiripple in passband (flat in stopband) → minimize maximum error

$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 T_N^2(\frac{\Omega}{\Omega_p})}$$



Chebyshev polynomial of order N →

$$T_N(\Omega) = \begin{cases} \cos(N\cos^{-1}\Omega) & |\Omega| \le 1 \\ \cosh(N\cosh^{-1}\Omega) & |\Omega| > 1 \end{cases}$$

Contd.,

$$T_N(x) = \cos(Nt)\big|_{x=\cos(t)}$$

$$T_0(x) = \cos(0) = 1,$$

$$T_1(x) = \cos(t)\big|_{x=\cos(t)} = x,$$

$$T_2(x) = \cos(2t) = 2\cos^2(t) - 1\big|_{x=\cos(t)} = 2x^2 - 1,$$

$$\ldots$$

$$T_{N+1}(x) = 2xT_N(x) - T_{N-1}(x)$$

Chebyshev polynomial

$$T_N(x) = \begin{cases} \cos(N \cos^{-1} x), & |x| \leq 1 \\ \cosh(N \cosh^{-1} x), & |x| > 1 \end{cases}$$

$$T_N(x) = 2xT_{N-1}(x) - T_{N-2}(x), \qquad N \geq 2$$

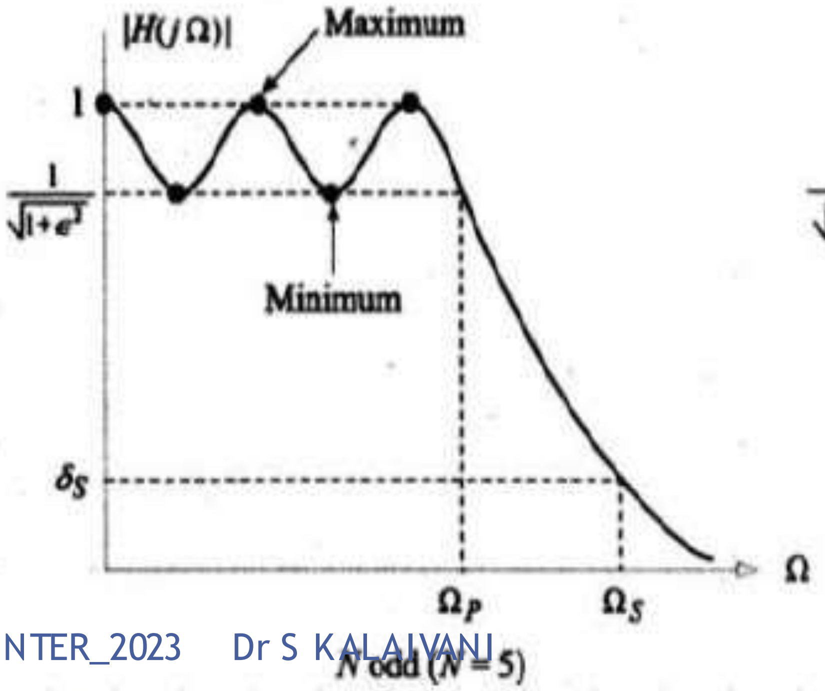
$$T_0(x) = 1 \quad \text{and} \quad T_1(x) = x$$

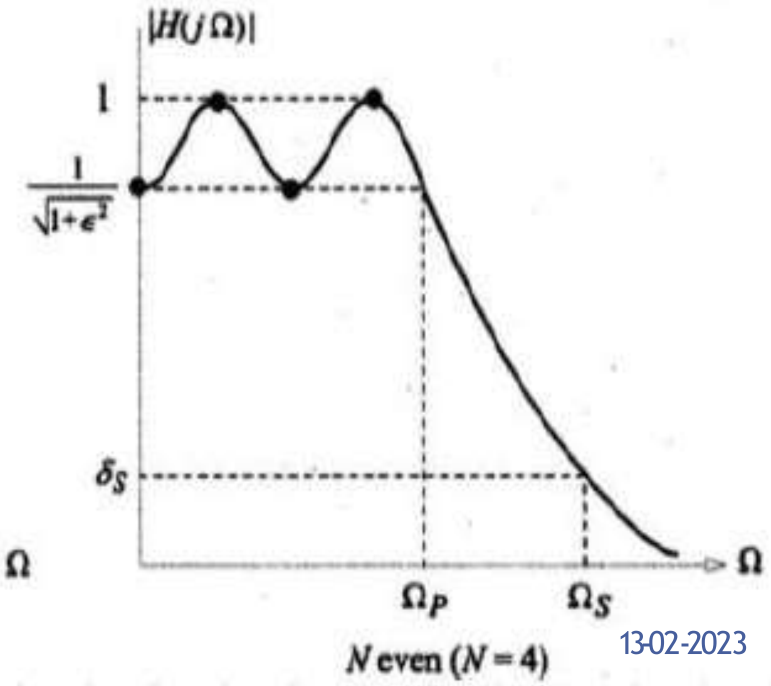
Table 4.3 The first five Chebyshev polynomials

N	$T_N(x)$
0	1
1	x
2	$2x^2 - 1$
3	$4x^3 - 3x$
4	$8x^4 - 8x^2 + 1$
5	$16x^5 - 20x^3 + 5x$

Magnitude Response

$$|H(j\Omega)| = \frac{1}{\left[1+\epsilon^2 T_N^2\left(\frac{\Omega}{\Omega_P}\right)\right]^{\frac{1}{2}}}$$





Class work_1

- For a fifth order chebychev low-pass filter with a passband of 1KHz and a passband ripple of 1 dB, What is the attenuation of this filter in dB at f=1 KHz and f=2 Khz?

Sol,

The magnitude frequency response of a lowpass Chebyshev I filter is

$$|H(j\Omega)| = \frac{1}{\left[1+\epsilon^2 T_N^2\left(\frac{\Omega}{\Omega_P}\right)\right]^{\frac{1}{2}}}$$

With $N = 5$, we get

$$|H(j\Omega)| = \frac{1}{\left[1+\epsilon^2 T_5^2\left(\frac{\Omega}{2\pi\times10^3}\right)\right]^{\frac{1}{2}}}$$

Given $K_P = -1$.

Since, $K_P = 20\log\left(\frac{1}{\sqrt{1+\epsilon^2}}\right)$, we get

$$20\log\left[\frac{1}{\sqrt{1+\epsilon^2}}\right] = -1$$

$$\Rightarrow \qquad \epsilon = 0.50885$$

$$\text{We know that,} \quad T_5(x) = 16x^5 - 20x^3 + 5x$$

$$\Rightarrow \quad T_5\left(\frac{\Omega}{2\pi\times10^3}\right) = 16\left(\frac{\Omega}{2\pi\times10^3}\right)^5 - 20\left(\frac{\Omega}{2\pi\times10^3}\right)^3 + 5\left(\frac{\Omega}{2\pi\times10^3}\right)$$

Hence,

$$|H(j\Omega)| = \frac{1}{\left[1+(0.50885)^2\left(16\left(\frac{\Omega}{2\pi\times10^3}\right)^5 - 20\left(\frac{\Omega}{2\pi\times10^3}\right)^3 + 5\left(\frac{\Omega}{2\pi\times10^3}\right)\right)^2\right]^{\frac{1}{2}}}$$

Hence,

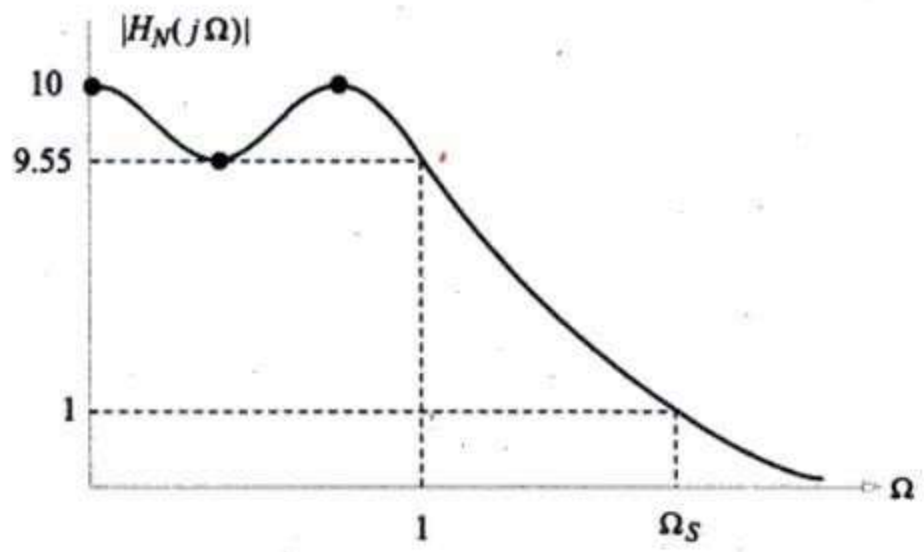
$$A_1 = -20\log|H(j\Omega)|_{\Omega=2\pi\times10^3} = 1\text{ dB}$$

and

$$A_2 = -20\log|H(j\Omega)|_{\Omega=2\pi\times2\times10^3} = 45.31\text{ dB}$$

Class work_2

- Find the order N, the value of the ripple factor ϵ, and the normalized passband attenuation of the Chebychev low pass filter for the given figure.



Sol.,

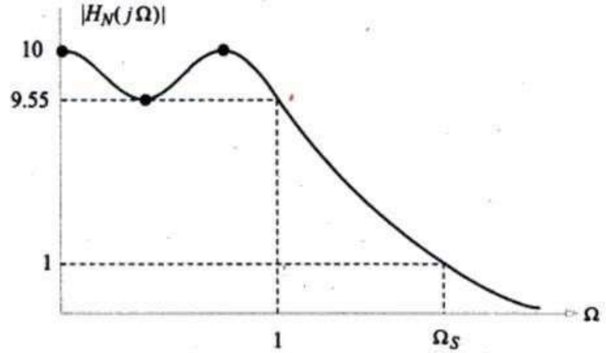
$$\frac{1}{\sqrt{1+\epsilon^2}} = 0.955$$
$$\epsilon = 0.3016$$

$$|H_N(j\Omega)| = \frac{1}{\left[1+\epsilon^2\, T_N^2(\Omega)\right]^{\frac{1}{2}}}$$
$$20\log|H_N(j\Omega)| = -10\log\left[1+\epsilon^2\, T_N^2(\Omega)\right]$$

Attenuation in dB is

$$A = -20\log|H_3(j\Omega)| \qquad (\because\ N=3)$$

$$A = 10\log\left[1+\epsilon^2\, T_3^2(\Omega)\right]$$



N=3

$$T_3(\Omega) = 4\Omega^3 - 3\Omega$$
$$A = 10\log\left[1+\epsilon^2\left(4\Omega^3-3\Omega\right)^2\right]$$

Letting $\Omega = 1$ and $\epsilon = 0.3016$, we get the passband attenuation,

$$A_P = 0.4 \text{ dB}$$