9. Find standard basis vector/vectors that can be added to the following set of vectors to produce a basis for R^3 and R^4.

(i) $\mathbf{v}_1 = (1, -1, 0)$, $\mathbf{v}_2 = (3, 1, -2)$

(ii) $\mathbf{v}_1 = (1, -2, 5, -3)$, $\mathbf{v}_2 = (2, 3, 1, -4)$

[**Ans.:** one possible basis (i) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1\}$ (ii) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3, \mathbf{e}_4\}$]

2.12 COORDINATE VECTOR RELATIVE TO A BASIS

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V then any vector $\mathbf{v}$ in V can be expressed as

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$$

The scalars $k_1, k_2, \ldots, k_n$ are called the coordinates of $\mathbf{v}$ relative to the basis S and the vector $(k_1, k_2, \ldots, k_n)$ in R^n is called the coordinate vector of $\mathbf{v}$ relative to S. This vector is denoted by

$$(\mathbf{v})_S = (k_1, k_2, \ldots, k_n)$$

Note: The coordinate vectors depend on the order in which the basis vectors are written. If the order of the basis vectors is changed, a corresponding change of order occurs in the coordinate vectors.

Example 1: Find the coordinate vector of v relative to the basis S.

(i) $\mathbf{v} = (1, 1)$; $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ where $\mathbf{v}_1 = (2, -4)$, $\mathbf{v}_2 = (3, 8)$

(ii) $\mathbf{v} = (5, -12, 3)$; $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (-4, 5, 6)$, $\mathbf{v}_3 = (7, -8, 9)$

Solution: (i) Let

$$
\begin{aligned}
(\mathbf{v})_S &= (k_1, k_2)\\
\mathbf{v} &= k_1\mathbf{v}_1 + k_2\mathbf{v}_2\\
(1, 1) &= k_1(2, -4) + k_2(3, 8)\\
&= (2k_1 + 3k_2, -4k_1 + 8k_2)
\end{aligned}
$$

Equating corresponding components,

$$
\begin{aligned}
2k_1 + 3k_2 &= 1\\
-4k_1 + 8k_2 &= 1
\end{aligned}
$$

Solving these equations,

$$k_1 = \frac{5}{28},\ k_2 = \frac{3}{14}$$

Hence, $(\mathbf{v})_S = \left(\frac{5}{28}, \frac{3}{14}\right)$

(ii) Let

$$
\begin{aligned}
(\mathbf{v})_S &= (k_1, k_2, k_3)\\
\mathbf{v} &= k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3\\
(5, -12, 3) &= k_1(1, 2, 3) + k_2(-4, 5, 6) + k_3(7, -8, 9)\\
&= (k_1 - 4k_2 + 7k_3, 2k_1 + 5k_2 - 8k_3, 3k_1 + 6k_2 + 9k_3)
\end{aligned}
$$

Equating corresponding components,

$$\begin{aligned} k_1 - 4k_2 + 7k_3 &= 5 \\ 2k_1 + 5k_2 - 8k_3 &= -12 \\ 3k_1 + 6k_2 + 9k_3 &= 3 \end{aligned}$$

Solving these equations,

$$k_1 = -2,\ k_2 = 0,\ k_3 = 1$$

Hence, $(\mathbf{v})_S = (-2, 0, 1)$

Example 2: Find the coordinate vector of $\mathbf{p} = 2 - x + x^2$ relative to the basis $S = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ where $\mathbf{p}_1 = 1 + x$, $\mathbf{p}_2 = 1 + x^2$, $\mathbf{p}_3 = x + x^2$.

Solution: Let

$$\begin{aligned} (\mathbf{p})_S &= (k_1, k_2, k_3) \\ \mathbf{p} &= k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\mathbf{p}_3 \\ 2 - x + x^2 &= k_1(1+x) + k_2(1+x^2) + k_3(x+x^2) \\ &= (k_1 + k_2) + (k_1 + k_3)x + (k_2 + k_3)x^2 \end{aligned}$$

Equating corresponding coefficients,

$$\begin{aligned} k_1 + k_2 \qquad &= 2 \\ k_1 + \qquad k_3 &= -1 \\ k_2 + k_3 &= 1 \end{aligned}$$

Solving these equations,

$$k_1 = 0,\ k_2 = 2,\ k_3 = -1$$

Hence, $(\mathbf{p})_S = (0, 2, -1)$

Example 3: Find the coordinate vector of A relative to the basis $S = \{A_1, A_2, A_3, A_4\}$, where

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix},\ A_1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution: Let $(A)_S = (k_1, k_2, k_3, k_4)$

$$A = k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4$$

$$\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} = k_1\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + k_2\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + k_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + k_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -k_1 + k_2 & k_1 + k_2 \\ k_3 & k_4 \end{bmatrix}$$

Equating corresponding components,

$$\begin{aligned} -k_1 + k_2 \qquad\qquad &= 2 \\ k_1 + k_2 \qquad\qquad &= 0 \\ k_3 \quad\; &= -1 \\ k_4 &= 3 \end{aligned}$$

Solving these equations,

$$k_1 = -1, k_2 = 1, k_3 = -1, k_4 = 3$$

Hence, $(A)_S = (-1, 1, -1, 3)$

Example 4: The vectors $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (0, 1, 2)$, $\mathbf{v}_3 = (3, 0, -1)$ form a basis of V. Let $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $S_2 = \{\mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_1\}$ are different orderings of these vectors. Determine the vector $\mathbf{v}$ in V having following coordinate vectors.

(i) $(\mathbf{v})_{S_1} = (3, -1, 8)$

(ii) $(\mathbf{v})_{S_2} = (3, -1, 8)$

Solution: (i) $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $k_1 = 3$, $k_2 = -1$, $k_3 = 8$

$$\begin{aligned} \mathbf{v} &= k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 \\ &= 3(1, -1, 1) - 1(0, 1, 2) + 8(3, 0, -1) \\ &= (3 + 24, -3 - 1, 3 - 2 - 8) \\ \mathbf{v} &= (27, -4, -7) \end{aligned}$$

(ii) $S_2 = \{\mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_1\}$ and $k_1 = 3$, $k_2 = -1$, $k_3 = 8$

$$\begin{aligned} \mathbf{v} &= k_1\mathbf{v}_3 + k_2\mathbf{v}_2 + k_3\mathbf{v}_1 \\ &= 3(3, 0, -1) - 1(0, 1, 2) + 8(1, -1, 1) \\ &= (9 + 8, -1 - 8, -3 - 2 + 8) \\ \mathbf{v} &= (17, -9, 3) \end{aligned}$$

In this example we observe that on changing the order of vectors in the basis, we get two different vectors in V corresponding to same coordinate vectors.

Coordinate Matrices

If $(\mathbf{v})_S = (k_1, k_2, \ldots, k_n)$ is the coordinate vector of $\mathbf{v}$ relative to the basis S then the coordinate matrix of $\mathbf{v}$ relative to the basis S is defined as

$$[\mathbf{v}]_S = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$$