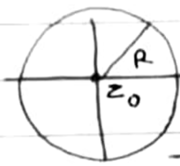
Taylor's Series :-

If $f(z)$ is analytic at all points inside a circle C, centre at z_0 and radius R then $f(z)$ can be represented as power series at all points inside C as



$|z-z_0|<$

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \cdots \infty$$

or $$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad |z-z_0|<R$$

where $a_n = \dfrac{f^{(n)}(z_0)}{n!}$

It is known as Taylor's series of $f(z)$ about the point $z=z_0$.

* when $z_0=0$ then

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \cdots + \frac{f^{(n)}(0)}{n!}z^n + \cdots$$

or $f(z) = a_n z^n$ is known as Maclaurin's series

Standard Maclaurin's series :-

(1) $$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots \infty, \quad |z|<\infty$$

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(2) $\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{2n+1!}$

$= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \infty \quad , |z| < \infty$

(3) $\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{2n!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} \cdots \infty, \ |z| < \infty$

(4) $\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \infty, \ |z| < \infty$

(5) $\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{n!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \infty, \ |z| < \infty$

(6) $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots + z^n + \cdots \quad |z| < 1$

(7) $\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n = 1 - z + z^2 - z^3 + \cdots \infty \quad , |z| < 1$

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Ex 1 Expand the function $f(z) = \log(1+z)$ in taylor series about $z_0 = 1$.

Sol^n Taylor series around $z = z_0$ is given by

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots \infty$$

$$f(z) = f(1) + \frac{f'(1)}{1!}(z-1) + \frac{f''(1)}{2!}(z-1)^2 + \cdots \infty \quad \text{——} (1)$$

Now

$$f(z) = \log(1+z)$$

$$f(1) = \log 2$$

$$f'(z) = \frac{1}{1+z}, \quad f'(1) = \frac{1}{2}$$

$$f''(z) = \frac{-1}{(1+z)^2}, \quad f''(1) = -\frac{1}{4}$$

$$f'''(z) = \frac{2}{(1+z)^3}, \quad f'''(1) = \frac{2}{8} = \frac{1}{4}$$

from (1)

$$f(z) = \log 2 + \frac{1}{2}\frac{(z-1)}{1!} - \frac{1}{4}\frac{(z-1)^2}{2!} + \frac{1}{4}\frac{(z-1)^3}{3!} \cdots$$

required Taylor's series

(2) Expand $f(z) = \sin z$, in Taylor's series about point $z=0$.

Solⁿ $z_0 = 0$.

$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \infty$$

where $a_n = \dfrac{f^{(n)}(z_0)}{n!} = \dfrac{f^{(n)}(0)}{n!}$

$f(z) = \sin z$, $f'(z) = \cos z$ $f''(z) = -\sin z$, $f'''(z) = -\cos z$

$a_0 = f(0) = \sin 0 = 0$ $\boxed{a_0 = 0}$

$$a_1 = \frac{f'(0)}{1!} = 1$$

$$a_2 = \frac{f''(0)}{2!} = 0$$

$$a_3 = \frac{f'''(0)}{3!} = -1$$

$$\boxed{f(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \infty}$$

Q expand e^{2z} in Taylor's series in taylor's series about point $z = 2i$.

Sol^n:- $f(z) = e^{2z}$, $z_0 = 2i$

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \frac{f'''(z_0)}{3!}(z - z_0)^3 + \cdots \infty$$

$f(z) = e^{2z}$ $f'(z) = 2e^{2z}$ $f''(z) = 4e^{2z}$ $f'''(z) = 8e^{2z}$

Now $f(z_0) = e^{4i}$

$f'(z_0) = 2e^{4i}$.

$f''(z_0) = 4e^{4i}$

$f'''(z_0) = 8e^{4i}$

$$\cancel{f(z) = e^{4i} + \frac{2e(z-2i)}{1!} e^{4i} + 4e^{8i}}$$

$$f(z) = e^{4i} + \frac{2e^{4i}}{1!}(z - 2i) + 4e^{4i}\frac{(z-2i)^2}{2!} + 8e^{4i}\frac{(z-2i)^3}{3!} + \cdots \infty$$

required Taylor's series

Q: Obtain Taylor series of $\cosh z$ about point $z_0 = \pi i$. Also find radius of convergence.

Hint

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

$$= \frac{e^{(z-\pi i)} \cdot e^{\pi i} + e^{-(z-\pi i)} e^{-\pi i}}{2}$$

$$= \frac{-e^{z-\pi i} - e^{-(z-\pi i)}}{2}$$

$$= \sum_{n=0}^{\infty} \frac{(z-\pi i)^n}{2n!}$$

radius of conv.

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \infty$$

Q → Expand $f(z) = \frac{1}{z}$ as Taylor's series about the point $z_0 = 1$. Also determine the region of convergence and radius of convergence.

Sol^n

$f(z) = \frac{1}{z}$ $\qquad f(1) = 1$

$f'(z) = -\frac{1}{z^2}$ $\qquad f'(1) = -1$

$f''(z) = \frac{2}{z^3}$ $\qquad f''(z) = 2!$

⋮

$f^{(n)}(z) = \frac{(-1)^n n!}{z^{n+1}}$ $\qquad f^{(n)}(1) = (-1)^n n!$

$$f(z) = f(z_0) + f'(z_0)\frac{(z-z_0)}{1!} + f''(z_0)\frac{(z-z_0)^2}{2!} + \cdots \infty$$

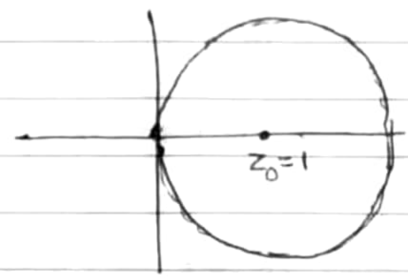
$$\frac{1}{z} = 1 - (z-1) + (z-1)^2 + \cdots + (-1)^n (z-1)^n + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n (z-1)^n$$

<u>Region of convergence</u>:-

$f(z)$ is not analytic at $z=0$

region of conv. is
$|z-1|<1$



and <u>radius of conv. is $R=1$.</u>

or $a_n = (-1)^n \quad a_{n+1} = (-1)^{n+1}$

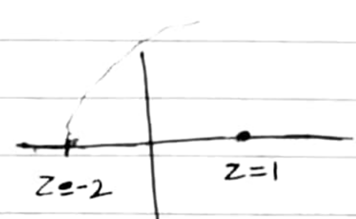
$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = 1$$

<u>Ex</u>:- Expand $f(z) = \frac{1}{z+2}$ in taylor series around $z=1$ also find region of convergence and radius of convergence

Ans $f(z) = \frac{1}{3} - \frac{1}{9}(z-1) + \frac{2}{27}\frac{(z-1)^2}{2!} - \frac{2}{27}\frac{(z-1)^3}{3!} + \cdots \infty$

Region of convergence
$|z-1|<3$

radius $= 3$



Partial fraction method

Another Method :-

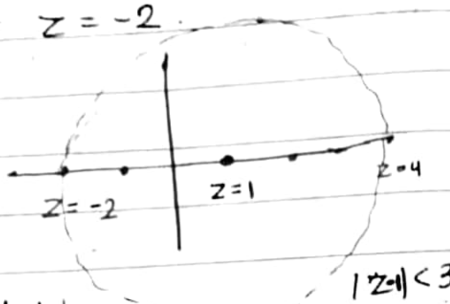
$(1+z)^{-1} = 1 - z + z^2 - z^3 + z^4 - - \infty$

$(1-z)^{-1} = 1 + z + z^2 + - - -$

Valid $|z| < 1$

$$f(z) = \frac{1}{z+2}$$

we need to expand $f(z)$ about $z = 1$
or in powers of $(z-1)$
$f(z)$ has singularity at $z = -2$.



$$f(z) = \frac{1}{z+2}$$

Let : $z - 1 = t \Rightarrow z = t + 1$

$$f(z) = \frac{1}{t+3}$$

$$= \frac{1}{3\left(1+\frac{t}{3}\right)}$$

$$= \frac{1}{3}\left(1+\frac{t}{3}\right)^{-1}, \quad \left|\frac{t}{3}\right| < 1 \Rightarrow |t| < 3$$

$$= \frac{1}{3}\left(1 - \frac{t}{3} + \frac{t^2}{3^2} - \frac{t^3}{3^3} + - - - \infty\right)$$

$$= \frac{1}{3} - \frac{t}{3^2} + \frac{t^2}{3^3} - \frac{t^3}{3^4} + - - - \infty$$

$$= \frac{1}{3} - \frac{(z-1)}{3^2} + \frac{(z-1)^2}{3^3} - \frac{(z-1)^3}{3^4} + - - \infty,$$

valid in $|z-1| < 3$

region of conv. $|z-1| < 3$
radius of conv $= 3$

Q Obtain Taylor's series expansion of the function $f(z) = \dfrac{z}{(z-3)(z-1)^2}$ about the point $z=-1$.

$\underline{\text{sol}^n}$ $f(z) = \dfrac{z}{(z-3)(z-1)^2}$

$$\frac{z}{(z-3)(z-1)^2} = \frac{A}{(z-3)} + \frac{B}{(z-1)} + \frac{C}{(z-1)^2}$$

$$z = A(z-1)^2 + B(z-1)(z-3) + C(z-3)$$

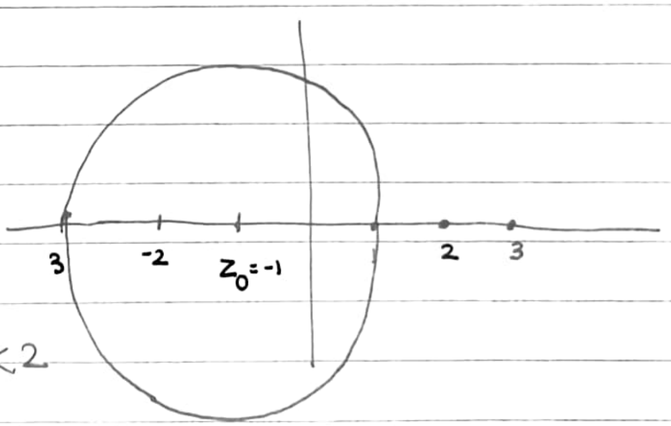
$A = -1/4$, $B = 1/4$ $C = 1/2$

$$f(z) = \frac{-1}{4(z-3)} + \frac{1}{4(z-1)} + \frac{1}{2(z-1)^2}$$

singularity $z = 3, 1$.

$z_0 = -1$

$$\boxed{\text{region of conv. is } |z+1| < 2}$$



put $z+1 = t$, ie $|t| < 2$

$$f(z) = \frac{-1}{4(t-4)} + \frac{1}{4(t-2)} + \frac{1}{2(t-2)^2}$$

$$f(z) = \frac{-1}{4(-4)}\left(1-\frac{t}{4}\right)^{-1} + \frac{1}{4\times(-2)}\left(1-\frac{t}{2}\right)^{-1} + \frac{1}{2\times -2}\left(1-\frac{t}{2}\right)^{-2}$$

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$$f(z) = \frac{1}{16}\left(1-\frac{t}{4}\right)^{-1} - \frac{1}{8}\left(1-\frac{t}{2}\right)^{-1} - \frac{1}{4}\left(1-\frac{t}{2}\right)^{-2}$$

$$f(z) = \frac{1}{16}\left(1+\frac{t}{4}+\frac{t^2}{16}+ \cdots \infty\right) - \frac{1}{8}\left(1+\frac{t}{2}+\frac{t^2}{4}+\cdots \infty\right)$$

$$-\frac{1}{4}\left(1+(-2)\left(\frac{-t}{2}\right)+\frac{(-2)(-2-1)}{2!}\left(\frac{-t}{2}\right)^2+\cdots\infty\right)$$

$$f(z) = \frac{1}{16}\left(1+\frac{t}{4}+\frac{t^2}{16}+\cdots\infty\right) - \frac{1}{8}\left(1+\frac{t}{2}+\frac{t^2}{4}+\cdots\infty\right)$$

$$-\frac{1}{4}\left(1+t+3.\frac{t^2}{4}+\cdots\infty\right)$$

$$f(z) = \left(\frac{1}{16}-\frac{1}{8}-\frac{1}{4}\right)+\left(\frac{1}{64}-\frac{1}{16}-\frac{1}{4}\right)t+\left(\frac{1}{256}-\frac{1}{32}-\frac{3}{16}\right)t^2+\cdots\infty$$

$$f(z) = -\frac{5}{16} + -\frac{19}{64}t + -\frac{55}{256}t^2 \cdots\cdots \infty$$

$$f(z) = \frac{-5}{16} - \frac{19}{64}(z+1) - \frac{55}{256}(z+1)^2 \cdots\cdots\infty$$

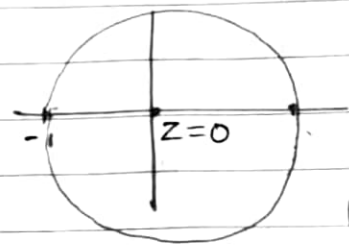
where $|z+1|<2$.

Q find Taylor series about the point $z=0$ and $z=1$ for the function $f(z) = \frac{z-1}{z+1}$

(i) about $z=0$

$$f(z) = \frac{z-1}{z+1} = 1 - \frac{2}{z+1}$$, $z=-1$ is singular Point

$$f(z) = 1 - \frac{2}{z+1}$$



$$= 1 - 2(1+z)^{-1}$$

$|z|<1$

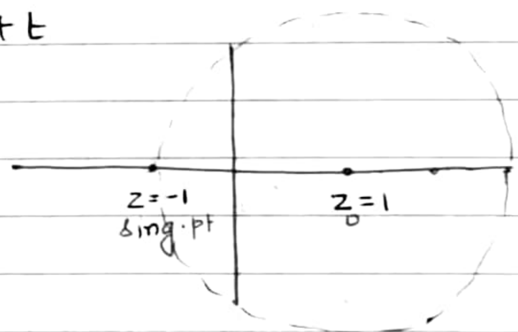
$$= 1 - 2(1 - z + z^2 - z^3 + z^4 - - - \infty) , \quad |z|<1$$

$$= -1 + z - z^2 + z^3 - z^4 + z^5 + - - \infty , \quad |z|<1$$

(ii) about point $z=1$ i.e in power of $(z-1)$

$$f(z) = 1 - \frac{2}{z+1}$$

Let $z-1=t \Rightarrow z = 1+t$



$$f(z) = 1 - \frac{2}{t+2}$$

$$= 1 - \frac{2}{2(1+\frac{t}{2})}$$

• $|z-1|<2$

$$= 1 - (1+\tfrac{t}{2})^{-1}$$, valid when $\left|\frac{t}{2}\right|<1$

$\Rightarrow |t|<2$

$$f(z) = 1 - \left(1+\frac{t}{2}\right)^{-1}, \quad |t|<2$$

$$= 1 - \left(1 - \frac{t}{2} + \frac{t^2}{4} - \frac{t^3}{8} + \cdots\right)$$

$$= \frac{t}{2} - \frac{t^2}{4} + \frac{t^3}{8} - \cdots \infty, \quad |t|<2$$

$$f(z) = \frac{(z-1)}{2} - \frac{(z-1)^2}{4} + \frac{(z-1)^3}{8} - \cdots \infty, \quad |z-1|<2$$

region of conv. $|z-1|<2$.

Q1- Find Taylor series expansion of complex fun.

$$f(z) = \frac{z+1}{(z-3)(z-4)} \quad \text{about } z=2.$$

Find region of convergence.

Soln:-
$$f(z) = \frac{z+1}{(z-3)(z-4)}$$

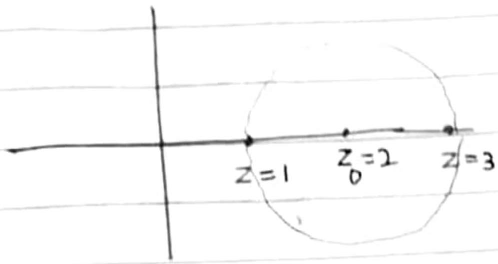
singular points $z=3$, $z=4$

we need to expand $f(z)$ in powers of $(z-2)$

Partial fraction

$$f(z) = \frac{z+1}{(z-3)(z-4)}$$

$$f(z) = \frac{5}{z-4} - \frac{4}{z-3}$$



$|z-2|<1$

$z-2 = t \Rightarrow z = t+2$

$$f(z) = \frac{5}{t+2-4} - \frac{4}{t+2-3}$$

$$f(z) = \frac{5}{t-2} - \frac{4}{t-1}$$

$$f(z) = \frac{-5}{2(1-\frac{t}{2})} + \frac{4}{(1-t)}$$

$$f(z) = \frac{-5}{2}\left(1-\frac{t}{2}\right)^{-1} + 4(1-t)^{-1}$$

$$= -\frac{5}{2}\left[1 + \frac{t}{2} + \frac{t^2}{4} + \frac{t^3}{8} + \cdots \infty\right] + 4(1+t+t^2+ \cdots \infty)$$

$$= \left(-\frac{5}{2}+4\right) + \frac{11}{4}t + \frac{27}{8}t^2 + \frac{59}{16}t^3 + \cdots \infty$$

$$= \frac{3}{2} + \frac{11}{4}t + \frac{27}{8}t^2 + \frac{59}{16}t^3 + \cdots \infty$$

$$f(z) = \frac{3}{2} + \frac{11}{4}(z-2) + \frac{27}{8}(z-2)^2 + \frac{59}{16}(z-2)^3 + \cdots \infty$$

$|z-2|<1$

region of conv. $= |z-2|<1$

radius " " $= 1$

Q: Find first three terms of Taylor series expansion of complex ~~variable~~ function $f(z) = \frac{1}{z^2+4}$ about $z = -i$. Find radius of convergence and region of convergence.

Sol^n

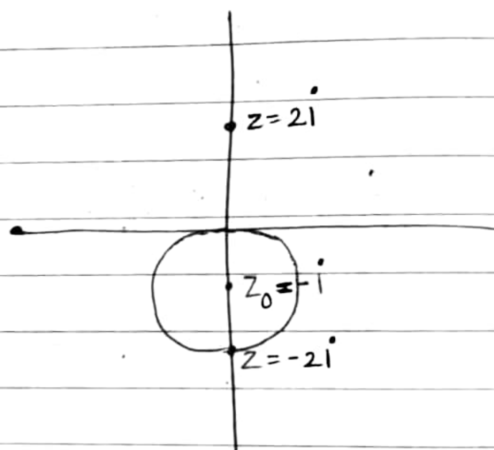
$$f(z) = \frac{1}{z^2+4}$$

$$f(z) = \frac{1}{(z+2i)(z-2i)}$$ singularities are $z = 2i, -2i$

$$f(z) = \frac{1}{4i}\left[\frac{1}{z-2i} - \frac{1}{z+2i}\right]$$

put $z + i = t$

$$f(z) = \frac{1}{4i}\left[\frac{1}{t-3i} - \frac{1}{t+i}\right].$$



$$= \frac{1}{4i}\left[\frac{1}{-3i\left(1-\frac{t}{3i}\right)} - \frac{1}{i\left(1+\frac{t}{i}\right)}\right]$$

$$= \frac{1}{12}\left(1 - \frac{t}{3i}\right)^{-1} + \frac{1}{4}\left(1 + \frac{t}{i}\right)^{-1}$$

$|z+i| < 1$
region of conv.
radius = 1

$$= \frac{1}{3} - \frac{2}{9i}t + \frac{1}{i^2}\frac{7}{27}t^2 + \text{-----} \infty.$$

$-\frac{1}{i} = i$

$$= \frac{1}{3} + \frac{i2}{9}t - \frac{7}{27}t^2 + \text{--} \infty$$

$$= \frac{1}{3} + \frac{i2}{9}(z+i) - \frac{7}{27}(z+i)^2 + \text{---}, \quad |z+i| < 1$$

Q:- Let $f(z) = \dfrac{3-i}{1-i+z}$ is expanded in a Taylor series with center $z_0 = 4-2i$. what is radius of convergence.

Q: ~~Find series solution o~~

Q. Expand $f(z) = \dfrac{z+5}{(z+1)(z+2)}$ in the region $|z|<1$.

Q Expand $f(z) = \dfrac{1}{z^2-z-6}$ about $z=1$.

Q Find series of $f(z) = \dfrac{z}{(z-1)(z-4)}$ in terms of $(z+3)$ valid for $|z+3|<4$.

Q Expand $f(z) = \dfrac{1}{4-3z}$ in taylor' series about $z_0 = 1+i$. Find region and radius of convergence.

Q:- $f(z) = \dfrac{1}{z(z-2i)}$ expand in taylor series about point $z=i$. Find region of convergence & radius of convergence