

Department of Mathematics
School of Advanced Sciences

Fall Semester 2022-2023

Continuous Assessment Test - II (October, 2022)

Course Code: BMAT201L

Course Title: Complex Variables & Linear Algebra

Max. Time: 90 minutes

Max. Marks: 50

Slot: A1+TA1+TAA1

Answer all the questions.

Write the answers in detail.

1. Find the Laurent's series of $f(z) = \frac{1}{(z-3)(z+1)}$ valid in the regions:

(a) $\mathcal{A} : 0 < |z-3| < 1$, (5 marks)

(b) $\mathcal{B} : 1 < |z-2| < 3$. (5 marks)

Solution: Note that $z = -1$ and 3 are poles of order 1, whereas $z = 0$ is a point of analyticity of f .

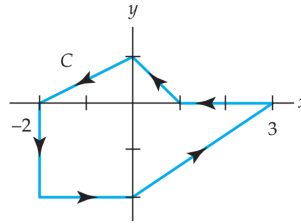
(a) Let $z - 3 = u$ for all points $z \in \mathcal{A} : 0 < |z - 3| < 1$. Then $z = u + 3$ and

$$\begin{aligned} f(z) &= \frac{1}{(z-3)(z+1)} = \frac{1}{4u} \left\{ 1 + \frac{u}{4} \right\}^{-1} \\ &= \frac{1}{4u} \sum_{n=0}^{\infty} (-1)^n \left(\frac{u}{4} \right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{u^{n-1}}{4^{n+1}} = \sum_{n=0}^{\infty} (-1)^n \frac{(z-3)^{n-1}}{4^{n+1}}. \end{aligned}$$

(b) Let $z - 2 = u$ for all points $z \in \mathcal{B} : 1 < |z - 2| < 3$. Then $z = u + 2$ and in \mathcal{B} , we have $1 < |u| < 3$ so that $|1/u| < 1$ and $|u/3| < 3$. Therefore,

$$\begin{aligned} f(z) &= \frac{1}{(u-1)(u+3)} = \frac{1}{4} \left\{ \frac{1}{u-1} - \frac{1}{u+3} \right\} = \frac{1}{4} \left\{ \frac{1}{u} \left\{ 1 - \frac{1}{u} \right\}^{-1} - \frac{1}{3} \left\{ 1 + \frac{u}{3} \right\}^{-1} \right\} \\ &= \frac{1}{4} \left\{ \frac{1}{u} \sum_{n=0}^{\infty} \left(\frac{1}{u} \right)^n - \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{u}{3} \right)^n \right\} = \frac{1}{4} \left\{ \sum_{n=0}^{\infty} \frac{1}{u^{n+1}} + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{u^n}{3^{n+1}} \right\} \\ &= \frac{1}{4} \left\{ \sum_{n=0}^{\infty} \frac{1}{(z-2)^{n+1}} + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(z-2)^n}{3^{n+1}} \right\}. \end{aligned}$$

- 2 (a). Employing Cauchy's integral formula, find $\oint_C f(z) dz$, where $f(z) = \frac{z-1}{z(z+i)}$ and C is as shown below: (5 marks)



Solution: By partial fractions,

$$f(z) = \frac{z-1}{z(z+i)} = -\frac{1}{i} \frac{1}{z} + \left(\frac{i+1}{i}\right) \frac{1}{z+i} = \frac{i}{z} + \frac{1-i}{z+i}.$$

Therefore, from Cauchy's integral formula, we get

$$\oint_C f(z) dz = i \oint_C \frac{1}{z} dz + (1-i) \oint_C \frac{1}{z+i} dz = 2\pi i (i+1-i) = 2\pi i.$$

- 2 (b). Using the residue theorem, find $\oint_\gamma \frac{1+4i}{(z-2)(z+2i)^2} dz$, where γ is the rectangle with sides $x = \pm 1, y = \pm \pi$. (5 marks)

Solution: Let $g(z) = \frac{1}{(z-2)(z+2i)^2}$. Note that $z = 2$ is a simple pole of g lying outside of γ , whereas $z = -2i$ is a double pole of g lying within γ . Also,

$$\begin{aligned} \text{Res}(g; -2i) &= \frac{1}{(2-1)!} \frac{d}{dz} \left\{ (z+2i)^2 f(z) \right\} \Big|_{z=-2i} = \frac{d}{dz} \left\{ \frac{1}{z-2} \right\} \Big|_{z=-2i} \\ &= -\frac{1}{(z-2)^2} \Big|_{z=-2i} = -\frac{1}{8i}. \end{aligned}$$

Therefore, by residue theorem,

$$\oint_C g(z) dz = 2\pi i \text{Res}(g; -2i) = I = 2\pi i \times \left\{ -\frac{1}{8i} \right\} = -\frac{\pi}{4}.$$

Therefore,

$$\oint_C f(z) dz = (1+4i)I = -\frac{(1+4i)\pi}{4}.$$

3. Compute $\int_{-\infty}^{\infty} \frac{\cos x}{x^2 - 10x + 9} dx$, through contour integration.

Solution: We compute

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 - 10x + 9} dx = \text{Real} \left\{ \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 - 10x + 9} dx \right\}.$$

Define $f(x) = \frac{e^{ix}}{x^2 - 10x + 9}$ so that $f(z) = \frac{e^{iz}}{z^2 - 10z + 9} = \frac{e^{iz}}{(z-1)(z-9)}$. Consider the contour $\gamma : \Gamma_R \cup [-R, R]$, where Γ_R is the semicircular arc $|z| = R = Re^{i\theta}$, $\theta = 0$ to π and R is sufficiently large. Then

$$\oint_{\gamma} f(z) dz = \int_{\Gamma_R} f(z) dz + \int_{-R}^R f(x) dx \quad (1)$$

Note that $z = 1$ and $z = 9$ are simple poles of f lying above the real axis within γ . Therefore, by residue theorem

$$\begin{aligned} \oint_{\gamma} f(z) dz &= 2\pi i \left\{ \text{Res}(f; 9) + \text{Res}(f; 1) \right\} = 2\pi i \left\{ \frac{e^{9i}}{8} - \frac{e^i}{8} \right\} \\ &= \frac{\pi i}{4} \left\{ (\cos 9 + i \sin 9) - (\cos 1 + i \sin 1) \right\} \\ &= \frac{\pi}{4} \left\{ (\sin 1 - \sin 9) + i(\cos 9 - \cos 1) \right\}. \end{aligned} \quad (2)$$

Using the appropriate upper bounds, we realize that $\int_{\Gamma_R} f(z) dz \rightarrow 0$ as $R \rightarrow \infty$. Therefore, applying the limit as $R \rightarrow \infty$ in (1) and using this and (2), we find that

$$\int_{-\infty}^{\infty} f(x) dx = \frac{\pi}{4} \left\{ (\sin 1 - \sin 9) + i(\cos 9 - \cos 1) \right\}$$

so that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 - 10x + 9} dx = \frac{\pi}{4} (\sin 1 - \sin 9).$$

4. Find the characteristic polynomial of $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. Using Caley-Hamilton's theorem,

express A^4 in terms of a quadratic matrix polynomial in A and find A^{-1} . What are the eigenvalues of A and A^2 and A^{-1} ?

Solution: The characteristic polynomial of A is

$$p(\lambda) = \lambda^3 - 4\lambda^2 - \lambda + 4.$$

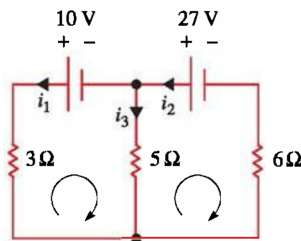
Then by Caley-Hamilton's theorem, $A^3 - 4A^2 - A + 4I = 0$. Hence

$$A^{-1} = \frac{1}{4}(-A^2 + 4A + I) = \frac{1}{4} \begin{pmatrix} -1 & 3 & -1 \\ 3 & -1 & -1 \\ -1 & -1 & 3 \end{pmatrix}.$$

$$A^4 = 17A^2 - 16I = \begin{pmatrix} -22 & -5 & -5 \\ -5 & -22 & -5 \\ -5 & -5 & -22 \end{pmatrix}.$$

Let λ_1, λ_2 and λ_3 be the eigenvalues of A . Then $\lambda_1 + \lambda_2 + \lambda_3 = \text{tr}(A) = 4$ and $\lambda_1\lambda_2\lambda_3 = \det(A) = -4$. By inspection, the triad $(1, -1, 4)$ satisfies these two conditions, and hence are the eigenvalues of A . Therefore, the eigenvalues of A^2 are 1, 1, 16 and the eigenvalues of A^{-1} are 1, -1, 1/4.

5 (a). The currents i_1, i_2 and i_3 in the network:



are described by the system of linear equations:

$$i_1 - i_2 + i_3 = 0,$$

$$3i_1 - 5i_3 = 10,$$

$$6i_2 + 5i_3 = 27.$$

Use *Gauss' elimination* method to solve the system for i_1, i_2 and i_3 . (5 marks)

Solution: The currents are $i_1 = 35/9, i_2 = 38/9$ and $i_3 = 1/3$

5 (b). Find the values of k such that the system

$$-3x_1 + 2x_2 - 2x_3 = 0,$$

$$x_1 - 2x_2 + 2x_3 = 0,$$

$$-2x_1 + 4x_2 + kx_3 = 0$$

has infinitely many nonzero solutions, and hence find all the corresponding nonzero solution vectors of the system. (5 marks)

Solution: The value of k is -4 and the nonzero solution vectors of the system are

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \alpha \neq 0.$$