Vellore – 632014, Tamil Nadu, India
DEPARTMENT OF MATHEMATICS
SCHOOL OF ADVANCED SCIENCES
FALL SEMESTER 2022-2023

CONTINUOUS ASSESSMENT TEST – I

Programme Name & Branch	: B.Tech. & ALL
Course Code	: BMAT201L
Course Name	: Complex Variables and Linear Algebra
Slot	: C1+TC1+TCC1
Date of the Examination	: 30-Aug-2022
Duration	**: 90 minutes** **Max. Marks : 50**

General instruction(s): Answer ALL questions (5x10=50 Marks)

1. Check whether the following functions can be the real parts of an analytic function $f(z) = u + iv$.

   a) $u = x^3 - y^3$
   b) $u = x^2 - y^2 + y$

   If so, determine the analytic function $f(z)$.

2. In a two dimensional fluid flow, if $\phi(x, y) = x^4 + y^4 - 6x^2y^2$ represents the velocity potential, find the corresponding stream function and also the complex potential.

3. Determine the region of the $w-$ plane into which the region bounded by the lines $x = 1$, $y = 1$, $x + y = 1$ is mapped by the transformation $w = z^2$.

4. If the points $\{0, 1, -2\}$ in the $z-$plane are mapped onto the points $\{-\frac{1}{2}, 0, \infty\}$ in the $w-$ plane respectively, then

   a) Find the corresponding bilinear transformation $w = f(z)$.
   b) Find the invariant points of this transformation.
   c) Find the image of the region $0 < x < \infty, 0 < y < \infty$ under this transformation.

5. Express the function $\mathrm{f(z)} = \frac{1}{(z+1)(z+2)^2}$ in Taylor series about

   a) the origin,
   b) the point $\mathrm{z} = 1$.

   Indicate the region of validity in each case.

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BMAT201L - CV & LA | SLOT-G_1+TG+TGG
CAT-I KEY

Q1) $u = x^3 - y^3 \Rightarrow u_{xx} = 6x,\ u_{yy} = -6y$

a) $u_{xx} + u_{yy} \neq 0$, u is not Harmonic. (4M)
Hence cannot be real part of $f(z)$.

b) $u = x^2 - y^2 + y \Rightarrow u_{xx} = 2,\ u_{yy} = -2$ (6M)
$u_{xx} + u_{yy} = 0$, u is Harmonic.
Hence u can be real part of $f(z)$.

$f'(z) = u_x + iv_x = u_x - iu_y$
$= 2x - i(-2y+1)$

By Milne Thomson method

$f'(z)\big|_{(z,0)} = 2z - i \Rightarrow f(z) = z^2 - iz + ic$

is the required Analytic function.

Q2) $\phi = x^4 + y^4 - 6x^2y^2$

$f'(z) = \phi_x + i\psi_x = \phi_x - i\phi_y$

$= (4x^3 - 12xy^2) - i(4y^3 - 12x^2y)$ (6M)

$f'(z) = 4z^3 \Rightarrow f(z) = z^4 + ic$
$= (x+iy)^4 + ic$

(4M) $= (x^4 - 6x^2y^2 + y^4) + (4x^3y - 4xy^3)i + ic$

$\therefore \psi(x,y) = 4x^3y - 4xy^3$

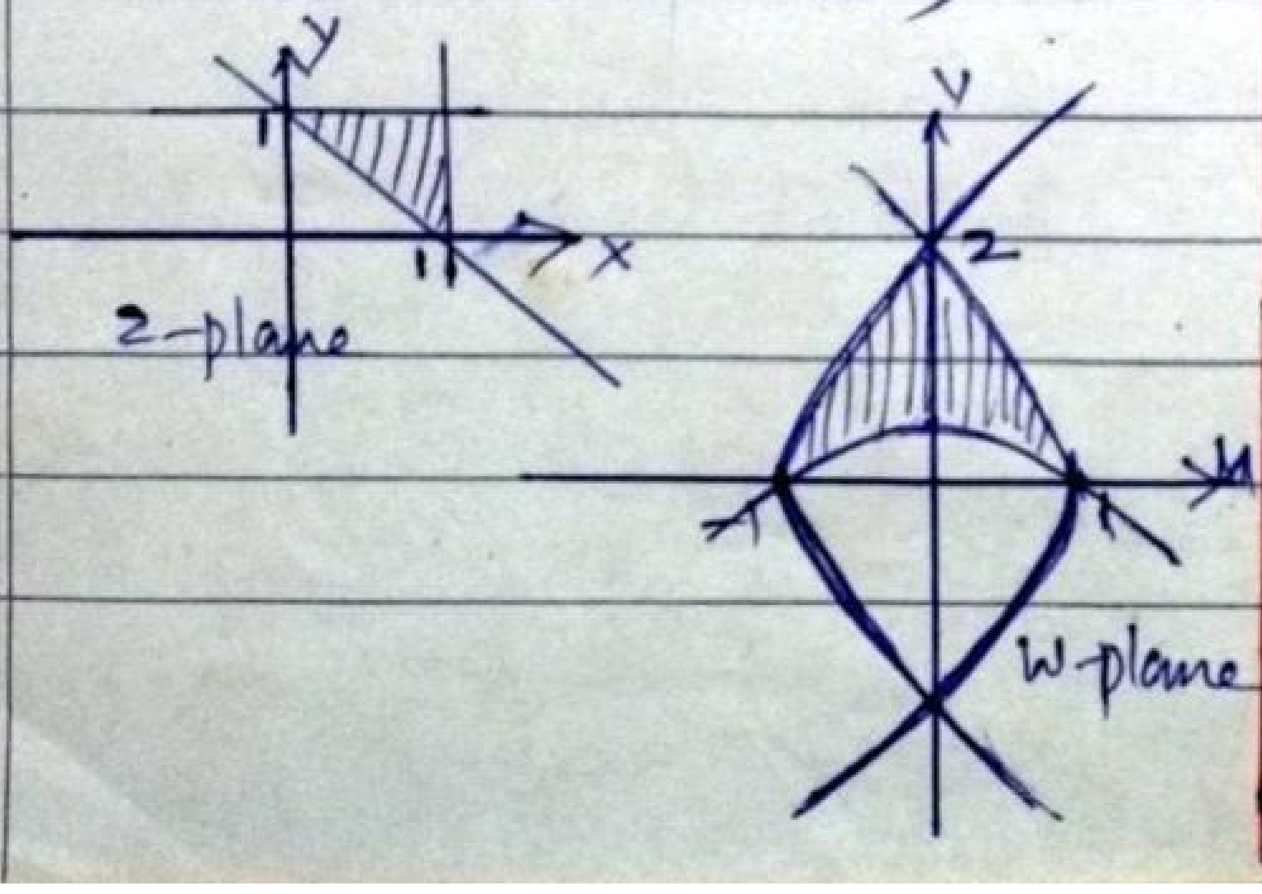
Q3) $w = z^2 = x^2 - y^2 + 2xyi$

$u = x^2 - y^2;\ v = 2xy$

$x = 1 \Rightarrow u = 1 - y^2,\ v = 2y$
$\Rightarrow v^2 = -4(u-1)$

$y = 1 \Rightarrow u = x^2 - 1$ and $v = 2x$
$\Rightarrow v^2 = 4(u+1)$

$x + y = 1 \Rightarrow u = x^2 - y^2 = (x+y)(x-y)$
$\Rightarrow u = x - y;\ v = 2xy$
$u^2 = (x-y)^2 = (x+y)^2 - 4xy$
$= 1 - 2v$
$\Rightarrow u^2 = -2(v - 1/2)$



Q4) $z\{0, 1, -2\} \leftrightarrow w\{-\frac{1}{2}, 0, \infty\}$

a) By Cross ratio property

$$\frac{(z-0)(1+2)}{(z+2)(1-0)} = \frac{(w+1/2)(0-\infty)}{(w-\infty)(0+1/2)} \quad (4M)$$

$w = \dfrac{z-1}{z+2}$ is the Bilinear Transformation

b) Put $w = z \Rightarrow z^2 + z + 1 = 0$ (2M)
$z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ are invariant Points.

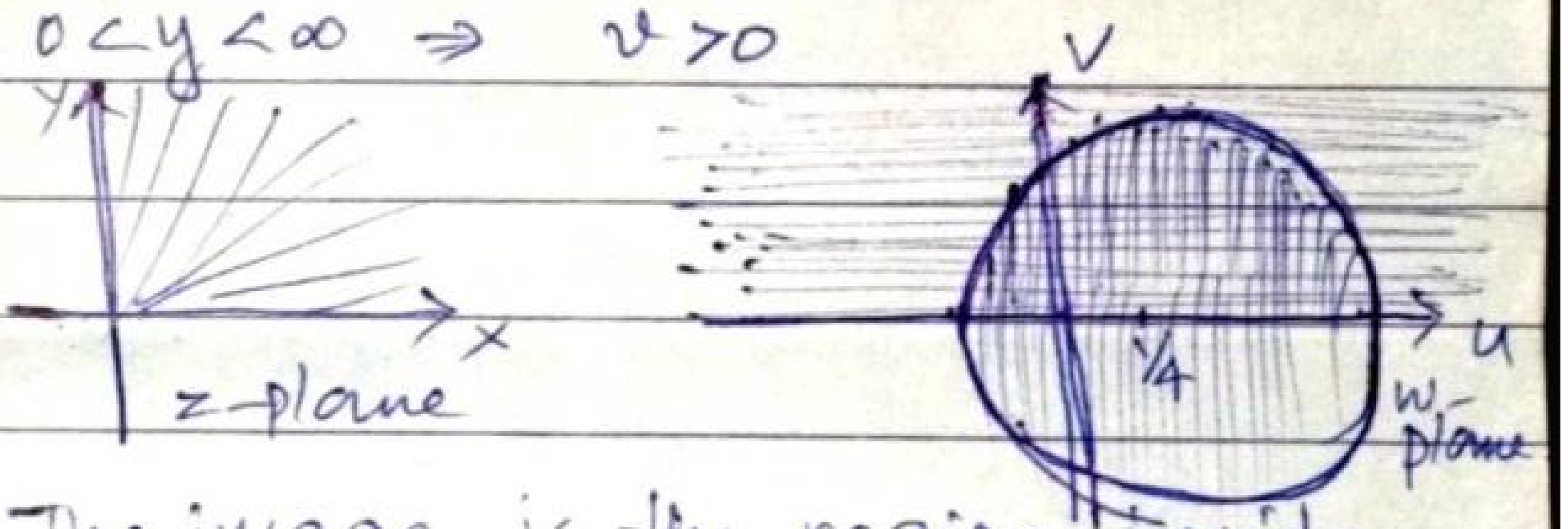
c) Rewriting the B.T. as (4M)

$$z = \frac{1+2w}{1-w} = \frac{1+2u+2vi}{1-u-vi}$$

$$x + iy = \frac{1+u-2u^2-2v^2}{(1-u)^2+v^2} + \frac{3v}{(1-u)^2+v^2}i$$

$$x = \frac{1+u-2u^2-2v^2}{(1-u)^2+v^2};\quad y = \frac{3v}{(1-u)^2+v^2}$$

$0 < x < \infty \Rightarrow (u - 1/4)^2 + v^2 < 9/16$
$0 < y < \infty \Rightarrow v > 0$



The image is the region inside the circle $(u - 1/2)^2 + v^2 = 9/16$ above the u-axis.

Q5) $$\frac{1}{(z+1)(z+2)^2} = \frac{1}{z+1} - \frac{1}{z+2} - \frac{1}{(z+2)^2} \quad (2M)$$

a) About Origin: (4M)

$f(z) = (1+z)^{-1} - \frac{1}{2}(1+z/2)^{-1} - \frac{1}{4}(1+z/2)^{-2}$

$= \{1 - z + z^2 - z^3 + \ldots\} - \frac{1}{2}\{1 - z/2 + z^2/4 - z^3/8 + \ldots\}$
$- \frac{1}{4}\{1 - 2\frac{z}{2} + 3\frac{z^2}{4} - 4\frac{z^3}{8} + \ldots\}$

$= \frac{1}{4} - \frac{z}{2} + \frac{11}{16}z^2 - \frac{13}{16}z^3 + \ldots \quad ; |z| < \infty$

b) Put $z - 1 = u$ (4M)

$f(z) = \frac{1}{u+2} - \frac{1}{u+3} - \frac{1}{(u+3)^2}$

$= \frac{1}{2}(1+u/2)^{-1} - \frac{1}{3}(1+u/3)^{-1} - \frac{1}{9}(1+u/3)^{-2}$

$= \frac{1}{2}\{1 - u/2 + u^2/4 - \ldots\} - \frac{1}{3}\{1 - u/3 + u^2/9 \ldots\}$
$- \frac{1}{9}\{1 - 2u/3 + 4u^2/9 - \ldots\}$

$= \frac{1}{18} - \frac{7}{108}u + \frac{11}{216}u^2 - \ldots \quad ; |P| < 2$

$= \frac{1}{18} - \frac{7}{108}(z-1) + \frac{11}{216}(z-1)^2 - \ldots$

valid in $|z-1| < 2$.