Vellore – 632014, Tamil Nadu, India
DEPARTMENT OF MATHEMATICS
SCHOOL OF ADVANCED SCIENCES
FALL SEMESTER 2022-2023

CONTINUOUS ASSESSMENT TEST – II

Programme Name & Branch : BTech
Course Code : BMAT201L
Course Name : Complex Variables and Linear Algebra
Slot : C2+TC2+TCC2
Date of the Examination : 12.10.22
Duration : 90 minutes **Max. Marks : 50**

General instruction(s):

Q. No	Question	Marks	Course Outcome (CO)	Bloom's Taxonomy (BL)
1.	Find the Laurent's series expansion of $f(z) = \frac{1}{(z^2+1)(z^2+2)}$ in the region $1 < \lvert z \rvert < \sqrt{2}$.	10	CO3	BL2
2.	a) Find the residue of $\frac{z^2-2z}{(z+1)^2(Z^2+4)}$ at all its poles. b) Evaluate $\int_C \frac{dz}{z^2+4}$, where C is $\lvert z-i \rvert = 2$ in the positive sense.	10	CO3	BL3
3.	Using contour integration, evaluate the real integral $\int_0^\infty \frac{dx}{(x^2+4)^3}$.	10	CO3	BL3
4.	Evaluate $A^8 - A^7 + 5A^6 - A^5 + A^4 - A^3 + 6A^2 + A - 2I$ if $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{bmatrix}$.	10	CO5	BL2
5.	Solve $Ax = b$, where $A = \begin{bmatrix} 4 & 1 & 1 & -2 \\ -4 & 0 & -1 & 4 \\ -12 & -1 & 4 & 5 \\ 0 & 0 & 14 & -7 \end{bmatrix}$ and $b = \begin{bmatrix} -7 \\ 8 \\ 0 \\ -49 \end{bmatrix}$ using Gauss elimination method.	10	CO5	BL3

1) Laurent's Series. Gn fn $f(z) = \dfrac{1}{(z^2+1)(z^2+2)}$

(1) $1 < |z| < \sqrt{2}$. In this annular region, we have

$1 < |z| \Rightarrow \left|\dfrac{1}{z}\right| < 1$ and.

$|z| < \sqrt{2} \Rightarrow \left|\dfrac{z}{\sqrt{2}}\right| < 1$

Hence $f(z) = \dfrac{1}{(z^2+1)(z^2+2)} = \dfrac{1}{(z^2+1)} - \dfrac{1}{z^2+2}$

$$= \frac{1}{z^2(1+1/z^2)} - \frac{1}{2(1+\frac{z^2}{2})}$$

$$= \frac{1}{z^2}\left(1+1/z^2\right)^{-1} - \frac{1}{2}\left(1+z^2/2\right)^{-1}$$

$$= \frac{1}{z^2}\left[1 - \frac{1}{z^2} + \frac{1}{z^4} + \cdots \frac{(-1)^n}{z^{2n}} + \cdots\right]$$

$$- \frac{1}{2}\left[1 - \frac{z^2}{2} + \frac{z^4}{4} - \frac{z^6}{8} \cdots \frac{(-1)^n z^{2n}}{2^n} + \cdots\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n+2}} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{n+1}} \cdot z^{2n}.$$

(2) $|z| > \sqrt{2}$

$\Rightarrow \dfrac{\sqrt{2}}{|z|} < 1 \Rightarrow \left|\dfrac{2}{z^2}\right| < 1.$

$$f(z) = \frac{1}{(z^2+1)(z^2+2)} = \frac{1}{z^2(1+1/z^2)} - \frac{1}{z^2(1+2/z^2)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n+2}} - \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{z^{2n+2}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (1-2^n)}{z^{2n+2}}$$ //

2) (a) Find the Residues of $f(z) = \dfrac{z^2-2z}{(z+1)^2(z^2+4)}$ at all its poles

Soln: The poles are

$z=-1$ is a double pole & $z=-2i, 2i$ are simple poles

$\therefore \text{Res}\{f(z) : z=-2i\} = \lim\limits_{z\to -2i} \cancel{(z+2i)} \dfrac{z^2-2z}{(z+1)^2\cancel{(z+2i)}(z-2i)}$

$= \dfrac{(-2i)^2-2(-2i)}{(-3-4i)(-4i)}$

$= \dfrac{1-i}{4-3i} = \dfrac{7-i}{25}$

Illy $\text{Res}\{f(z) : z=2i\}$ is $\dfrac{7+i}{25}$

Since $z=-1$ is a double pole.

$\text{Res}\{f(z) : z=-1\} = \dfrac{g'(-1)}{1!}$ where

$g(z) = \dfrac{z^2-2z}{z^2+4}$ which is analytic at $z=-1$

$g(-1) \neq 0$.

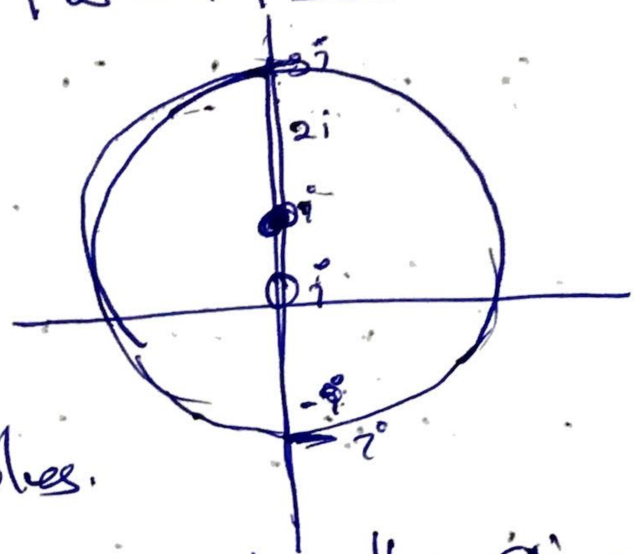
$g'(z) = \dfrac{2z^2+8z-8}{(z^2+4)^2}$;

$\text{Res}\{f(z) : z=-1\} = \dfrac{2(-1)^2+8(-1)-8}{((-1)^2+4)^2} = \dfrac{-14}{25}$ //

Q. b) Evaluate $\int_C \frac{dz}{z^2+4}$ where C is $|z-i|=2$ in the positive orientation.

Soln:- Gn Circle $|z-i|=2$

The poles of $\frac{1}{z^2+4}$ are $z=-2i$ and $z=2i$, ~~and~~ simples poles.



Also $z=2i$ lives inside the given Circle.

∴ By Cauchy's Residue theorem

$$\int_C \frac{dz}{z^2+4} = 2\pi i \{\text{Sum of the residues of } f(z)\}$$

$$= 2\pi i \cdot \frac{1}{4i} = \frac{\pi}{2}$$

$Res\{f(z) : z=2i\}$

$$= \lim_{z \to 2i} \cancel{(z-2i)} \frac{1}{(z+2i)\cancel{(z-2i)}}$$

$$= \frac{1}{4i}$$

3) Using Contour Integration, evaluate the Real Integral $\int_0^{\infty} \frac{dx}{(x^2+4)^3}$

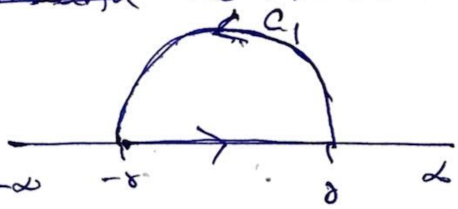
Soln: $\int_0^{\infty} \frac{dx}{(x^2+4)^3} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^3}$

($\because \frac{1}{(x^2+4)^3}$ is even)

$\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^3} = ?$

$\int_C \cancel{f(z)} \frac{dz}{(z^2+4)^3} = \int_{-r}^{r} \frac{dx}{(x^2+4)^3} + \int_{C_1} \frac{dz}{(z^2+4)^3}$

where C is the semicircle $|z| = r$, ~~and~~ with the interval $[-r, r]$ on the real axis



As $r \to \infty$ $\int_{C_1} \frac{dz}{(z^2+4)^3} = 0$

$\therefore \int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^3} = \int_C \frac{dz}{(z^2+4)^3}$

$\oint_C \frac{dz}{(z^2+4)^3} = 2\pi i \{\text{sum of the Residues}\}$

The poles are $z = -2i$ & $z = 2i$ of order three

~~and~~ Moreover $z = 2i$ lies in the upper half.

i.e. $\operatorname{Res}\{f(z) \text{ at } z = 2i\} = \frac{g''(2i)}{2!}$ where

$g(z) = \frac{1}{(z+2i)^3}$, ~~where~~ g is analytic at $z = 2i$ and $g(2i) \neq 0$

$\therefore \operatorname{Res} = \frac{\frac{12}{(4i)^5}}{2!} = \frac{6}{(4i)^5} = \frac{6}{1024 i} = \frac{3}{512 i}$

$$\int_C \frac{dz}{(z^2+4)^3} = 2\pi \cancel{i} \cdot \frac{3}{512\cancel{i}} = \frac{6\pi}{512}$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^3} = \frac{6\pi}{512}$$

$$\int_0^{\infty} \frac{dx}{(x^2+4)^3} = \frac{3\pi}{512} \quad //$$

4) Evaluate $A^8 - A^7 + 5A^6 - A^5 + A^4 - A^3 + 6A^2 + A - 2I$

if $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{bmatrix}$

Soln: Characteristic polynomial

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 & -2 \\ 2 & 5-\lambda & -4 \\ 3 & 7 & -5-\lambda \end{bmatrix}$$

$$= (1-\lambda)\left[(5-\lambda)(-5-\lambda)+28\right] - 2\left[2(-5-\lambda)+12\right] - 2\left[14-3(5-\lambda)\right]$$

$$= (1-\lambda)\left[\lambda^2+3\right] - 2\left[2-2\lambda\right] - 2\left[-1+3\lambda\right]$$

$$= \lambda^2 + 3 - \lambda^3 - 3\lambda - 4 + 4\lambda + 2 - 6\lambda$$

$$= -\lambda^3 + \lambda^2 - 5\lambda + 1.$$

By Cayley Hamilton theorem we have

$$A^3 - A^2 + 5A - I = 0$$

Gn $A^8 - A^7 + 5A^6 - A^5 + A^4 - A^3 + 6A^2 + A - 2I$

$$= A^5\underbrace{\left[A^3 - A^2 + 5A - I\right]}_{0} + A\left[A^3 - A^2 + 5A - I + A + 2\right] - 2I$$

$$= 0 + A\underbrace{\left[A^3 - A^2 + 5A - I\right]}_{0} + A^2 + 2A - 2I$$

$$= A^2 + 2A - 2I$$

$$= \begin{bmatrix} -1 & 2 & -4 \\ 4 & 9 & -12 \\ 8 & 20 & -12 \end{bmatrix}$$

5. Solve $Ax=b$, where $A = \begin{bmatrix} 4 & 1 & 1 & -2 \\ -4 & 0 & -1 & 4 \\ -12 & -1 & 4 & 5 \\ 0 & 0 & 14 & -7 \end{bmatrix}$

and $b = \begin{bmatrix} -7 \\ 8 \\ 0 \\ -49 \end{bmatrix}$

Solution:-

$$[A|b] = \left[\begin{array}{cccc|c} 4 & 1 & 1 & -2 & -7 \\ -4 & 0 & -1 & 4 & 8 \\ -12 & -1 & 4 & 5 & 0 \\ 0 & 0 & 14 & -7 & -49 \end{array}\right]$$

$$\left.\begin{array}{l} R_2 \to R_2 - (-1)R_1 \\ R_3 \to R_3 - (-3)R_1 \end{array}\right\} \Rightarrow \left[\begin{array}{cccc|c} 4 & 1 & 1 & -2 & -7 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 2 & 7 & -1 & -21 \\ 0 & 0 & 14 & -7 & -49 \end{array}\right]$$

$$R_3 \to R_3 - 2R_2 \quad \left[\begin{array}{cccc|c} 4 & 1 & 1 & -2 & -7 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 7 & -5 & -23 \\ 0 & 0 & 14 & -7 & -49 \end{array}\right]$$

$$R_4 \to R_4 - 2R_3 \quad \left[\begin{array}{cccc|c} 4 & 1 & 1 & -2 & -7 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 7 & -5 & -23 \\ 0 & 0 & 0 & 3 & -3 \end{array}\right]$$

Upper triangular system:

$$\begin{aligned} 4x_1 + x_2 + x_3 - 2x_4 &= -7 \\ x_2 \qquad + 2x_4 &= 1 \\ 7x_3 - 5x_4 &= -23 \\ 3x_4 &= -3 \end{aligned}$$

pivots = 4 = # unknowns

$\Rightarrow$ The system has unique solution.

$$\left\{\begin{array}{l} x_4 = -1 \\ 7x_3 - 5(-1) = -23 \Rightarrow x_3 = -4 \\ x_2 + 2(-1) = 1 \Rightarrow x_2 = 3 \\ 4x_1 + (3) - 4 - 2(-1) = -7 \\ \qquad \Rightarrow x_1 = -2 \end{array}\right.$$

$x = \begin{bmatrix} -2 \\ 3 \\ -4 \\ -1 \end{bmatrix}$ is the solution