VIT

Vellore Institute of Technology

(Deemed to be University under section 3 of UGC Act, 1956)

SCHOOL OF ADVANCED SCIENCES

Fall Semester 2023-2024

Continuous Assessment Test – I

Programme Name: B. Tech

Slot: E1+TE1

Course Name & code: BMAT101L & Calculus

Exam Duration: 90 Minutes **Maximum Marks:** 50

<u>Answer all the questions</u>

Q.No.	Question	Max. Marks
1.	a) State the Rolle's Theorem. b) For the following function $y(x) = \lvert\sin x\rvert$, verify that the hypotheses of Rolle's Theorem are satisfied on the interval $[0,2\pi]$, and find all values of x in the interval $(0,2\pi)$ that satisfy the conclusion of the theorem.	10
2.	Find the intervals in which the function given by $f(x) = x^4 - 8x^2 + 16$ is increasing, decreasing, concave up and concave down.	10
3.	Find the area enclosed by the region $\{(x,y): y^2 \leq 3x, 3x^2 + 3y^2 \leq 16\}$ using the method of integration. Also, draw a rough sketch of the region.	10
4.	Investigate the continuity of the function $f(x,y) = \frac{\sqrt[3]{x}\, y^2}{x+y^3}$ at origin and find first partial derivatives at orgin if exists.	10
5.	Use an appropriate form of the chain rule to find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ for $z = 3x - 2y$, $x = u + v\ln(u)$ and y=$u^2 - v\ln(v)$.	10

Fall semester 2023 - 2024

CAT-I

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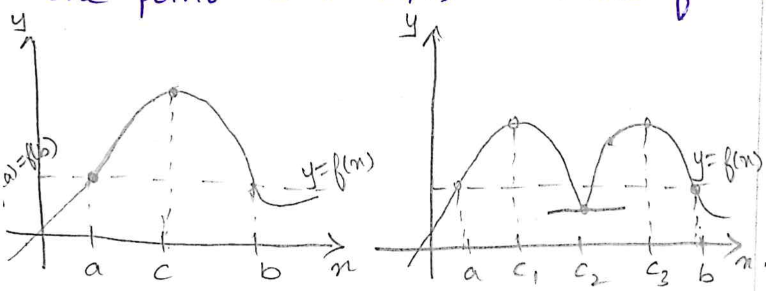
ANSWER KEY

Q1 a Rolle's Theorem statement:

Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a,b]$ and differentiable at every point of its interior (a,b). If $f(a) = f(b)$ then there is atleast one point c in (a,b) at which $f'(c) = 0$.



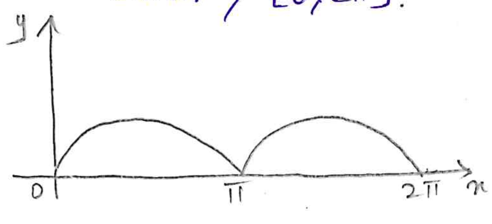
Q1 b Given function $y(x) = |\sin x|$ in the interval $[0, 2\pi]$

* $|\sin x|$ is continuous for any x but it is not differentiable at $x = \pi$ in $(0, 2\pi)$

Hence, it doesnot satisfy the hypothesis of Rolle's theorem.

But the function $y(x) = |\sin x|$ satisfy the hypothesis of the Rolle's theorem where x lies in the subinterval $(0,\pi)$ and $(\pi, 2\pi)$

Hence, finding the values of x is not possible for the given function $y(x) = |\sin x|$, $[0, 2\pi]$.



Q.2 Given $f(x) = x^4 - 8x^2 + 16$

$\Rightarrow f'(x) = 4x^3 - 16x$

$\Rightarrow f''(x) = 12x^2 - 16$

* Now, in the interval in which

if $f'(x)$ is +ve, then function is increasing

if $f'(x)$ is -ve, then function is decreas

if $f''(x)$ is +ve, then function is concave u

if $f''(x)$ is -ve, then function is concave do

* Now, $f'(x) = 4x^3 - 16x$

$= 4x(x-2)(x+2)$

$\Rightarrow$ in interval, $(-\infty, -2)$, $f'(-3) = -108 + 48 = -60 < 0$

$[-2, 0)$, $f'(-1) = -4 + 16 = 12$

$[0, 2)$, $f'(1) = 4 - 16 = -12 <$

$[2, \infty)$, $f'(3) = 180 - 48 = 60 >$

$\therefore$ In the interval

$(-\infty, -2)$, function $f(x)$ is decreasing

$[-2, 0)$, $f(x)$ is increasing

$[0, 2)$, $f(x)$ is decreasing

$[2, \infty)$, $f(x)$ is increasing.

$\rightarrow$ Now, $f''(x) = 12\left(x - \frac{2}{\sqrt{3}}\right)\left(x + \frac{2}{\sqrt{3}}\right)$

$\Rightarrow$ in interval, $\left(-\infty, \frac{-2}{\sqrt{3}}\right]$, $f''(-5) = 284 >$

$\left(-\frac{2}{\sqrt{3}}, \frac{2}{3}\right]$, $f''(0) = -16 < 0$

$\left(\frac{2}{\sqrt{3}}, \infty\right)$, $f''(5) = 284 > 0$

$\therefore$ In the interval

$\left(-\infty, -\frac{2}{\sqrt{3}}\right]$, $f(x)$ is concave up

$\left(\frac{-2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$, $f(x)$ is concave down

$\left(\frac{2}{\sqrt{3}}, \infty\right)$, $f(x)$ is concave up

Q.3 Given : $y^2 \le 3x$, $3x^2+3y^2 \le 16$

To find : Area of the region enclosed by the curves

$$y^2 = 3x$$

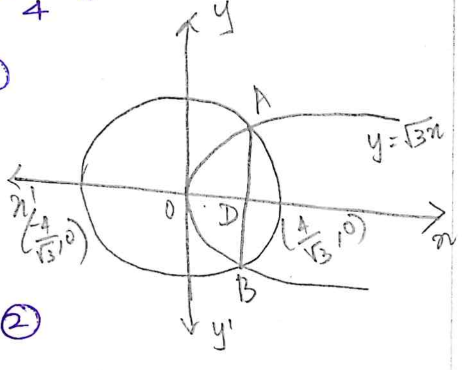
(or), $y^2 = 4 \times \frac{3}{4} x$

$\therefore y = \sqrt{3x} \to (1)$

$3x^2 + 3y^2 = 16$

$\therefore x^2 + y^2 = \left(\frac{4}{\sqrt{3}}\right)^2$

$\therefore y = \sqrt{\frac{16-3x^2}{3}} \to (2)$



omparing eqn (1) & (2), we get

$$3x = \frac{16-3x^2}{3}$$

or, $3x^2 + 9x - 16 = 0$

$$\therefore x = \frac{-9 \pm \sqrt{81+192}}{6}$$

$$= \frac{-9 \pm \sqrt{273}}{6}$$

x cannot be negative from the above figure.

$$\therefore x = \frac{-9+\sqrt{273}}{6} = a \text{ (let)}$$

Required area $= 2\left[\int_0^a \sqrt{3x}\, dx + \int_a^{4/\sqrt{3}} \sqrt{\frac{16}{3} - x^2}\, dx\right]$

$$= 2\left[\sqrt{3}\left\{\frac{x^{3/2}}{3/2}\right\}_0^a + \left\{\frac{x}{2}\sqrt{\frac{16}{3}-x^2} + \frac{16}{3}\sin^{-1}\left(\frac{x}{4/\sqrt{3}}\right)\right\}_a^{4/\sqrt{3}}\right]$$

$$= 2\left[\frac{2}{\sqrt{3}} a^{3/2} + \left(\frac{4}{\sqrt{3}} + \frac{1}{2}\right)\left(\sqrt{\frac{16}{3} - \frac{16}{3}}\right) + \frac{16}{3}\sin^{-1}\left(\frac{4/\sqrt{3}}{4/\sqrt{3}}\right) - \frac{a}{2}\sqrt{\frac{16}{3} - a^2} - \frac{16}{3}\sin^{-1}\left(\frac{4}{4/\sqrt{3}}\right)\right]$$

$$= \frac{4}{3} a^{3/2} + \left(\frac{32}{3} \times \frac{\pi}{2}\right) - a\sqrt{\frac{16}{3} - a^2} - \frac{16}{3}\sin^{-1}\left(\frac{\sqrt{3}a}{4}\right)$$

$$= \frac{4}{3} a^{3/2} + \frac{16\pi}{3} - a\sqrt{\frac{16}{3} - a^2} - \frac{16}{3}\sin^{-1}\left(\frac{\sqrt{3}a}{4}\right)$$

Hence, the area of the region is

$$\left\{\frac{4}{3} a^{3/2} + \frac{16\pi}{3} - a\sqrt{\frac{16}{3} - a^3} - \frac{16}{3}\sin^{-1}\left(\frac{\sqrt{3}a}{4}\right)\right\}$$

where $a = \frac{-9+\sqrt{273}}{6}$.

Q.4

$$\lim_{x,y \to 0} \frac{x^{1/3} y^2}{x + y^3}$$

Let $x^{1/3} = my$

$$\Rightarrow \lim_{x,y \to 0} \frac{my \cdot y^2}{m^3 y^3 + y^3} = \frac{m}{m^3+1}$$

The value of limit varies for different values of m. So limit doesnot exists. So the function is not continuous.

Q.5 Given $z = 3x - 2y$

$x = u + v \ln(u)$ and $y = u^2 - v \ln(v)$

To find : $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

$z_x = 3$, $z_y = -2$,

$x_u = 1 + \frac{v}{u}$, $x_v = \ln(u)$

$y_u = 2u$, $y_v = -(1 + \ln v)$

$$z_u = z_x \cdot x_u + z_y \cdot y_u$$
$$= 3\left(1 + \frac{v}{u}\right) - 4u$$

$$z_v = z_x x_v + z_y \cdot y_v$$
$$= 3\ln(u) + 2(1 + \ln v)$$