Fall-Semester 2023-24

Continuous Assessment Test – I

Programme Name & Branch: B.Tech.

Course Name & Code: Calculus (BMAT101L)

Exam Duration: 90 minutes | **Maximum Marks: 50**

Class Number: VL2023240106111 | **Slot: G1+TG1**

Answer All the Questions (5 × 10 = 50)

S. No.	Question	Max Marks	CO	BL
1.	Using the first and second derivative tests, discuss the monotonicity, relative extrema, concavity and the points of inflection of $f(x)=\frac{2x^3}{3}+\frac{x^2}{2}-x$. Given function $f(x)=\frac{2x^3}{3}+\frac{x^2}{2}-x$. $f'(x)=2x^2+x-1$ $f'(x)=0 \Rightarrow$ critical points $=-1, \frac{1}{2}$ $-\infty \quad -1 \quad \frac{1}{2} \quad \infty$ $f'(x) \quad + \quad - \quad +$ $f(x) \quad \uparrow \quad \downarrow \quad \uparrow$ $f''(x)=4x+1$ $f''(x)=4x+1$ $f''(\frac{1}{2})>0 \Rightarrow f(\frac{1}{2})=-\frac{7}{24}$ (L. minima) $=-0.292$ $f''(\frac{1}{2})<0 \Rightarrow f(-1)=\frac{5}{6}$ (L. maxima) $=0.8333$ $f''(x)=0 \Rightarrow x=-\frac{1}{4}$ and $f''(x)<0, \forall x<-\frac{1}{4}$ (concave down) $f''(x)>0, \forall x>-\frac{1}{4}$ (concave up) $\therefore x=-\frac{1}{4}$ is a point of inflection.	10	1	1

2.	State Rolle's theorem and Verify Rolle's theorem for the function $f(x)=e^x \sin x,\ 0 \le x \le \pi$. $\because$ e^x and $\sin x$ are Continuous for all x, therefore the product $e^x \sin x$ is Continuous in $0 \le x \le \pi$. $f'(x) = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$ exist in $0 < x < \pi \Rightarrow f'(x)$ is differentiable in $(0,\pi)$. $f(0) = e^0 \sin 0 = 0$ $f(\pi) = e^\pi \sin \pi = 0$. $\therefore$ f satisfies hypothesis of Rolle's theorem Thus there exists $c \in (0,\pi)$ satisfying $f'(c)=0 \Rightarrow$ $e^c(\sin c + \cos c) = 0 \Rightarrow e^c = 0$ or $\sin c + \cos c = 0$ $e^c = 0 \Rightarrow c = -\infty$ which is not meaningful here. $\Rightarrow \sin c = -\cos c \Rightarrow \frac{\sin c}{\cos c} = -1 \Rightarrow \tan c = -1 = \tan\frac{3\pi}{4}$ $\Rightarrow c = 3\pi/4$ is the required Point.	10	1	2
3.	Consider the region bounded by the graphs of $2x+3y=6$, $y=0$ and $x=0$. Find the volume of the solid formed by revolving this region about the y axis and the line $x=-2$.	10	1	3

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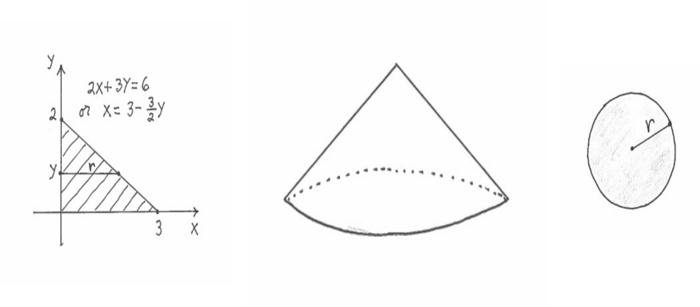
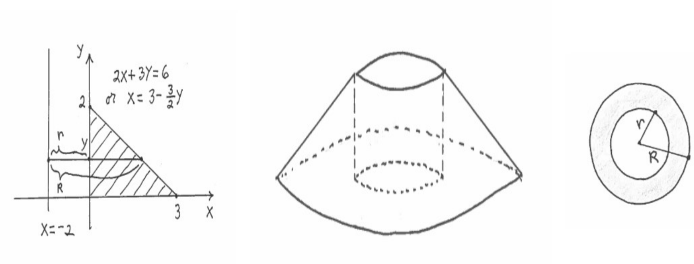
<td>SOLUTION 2: a.) IMPORTANT CHANGE: Because we are revolving the region about the y-axis, we must make slices perpendicular to the y-axis at y !!! This ensures that the slices are CIRCULAR. Here are a carefully labeled sketch of the region, a rough sketch of the resulting Solid of Revolution, and a circular cross-section at y. In this example, the cross section is called an annulus, a circular region of radius R with a smaller concentric circular region of radius r removed. It is IMPORTANT to mark ALL of y, r, and R in the sketch of the region !!!


The area of the circular cross-section is
$$A(y) = \pi r^2 = \pi(3 - \frac{3}{2}y)^2$$
Thus the total volume of this Solid of Revolution is
$$Volume = \int_0^2 \pi(3 - \frac{3}{2}y)^2 \, dy$$
SOLUTION 2: b.) Here are a carefully labeled sketch of the region, a rough sketch of the resulting Solid of Revolution, and a circular cross-section at x. In this example, the cross section is called an annulus, a circular region of radius R with a smaller concentric circular region of radius r removed. It is IMPORTANT to mark ALL of x, r, and R in the sketch of the region !!!


The area of the circular cross-section is
$$A(y) = \pi R^2 - \pi r^2 = \pi((3 - \frac{3}{2}y) - (-2))^2 - \pi(2)^2 = \pi(5 - \frac{3}{2}y)^2 - \pi(2)^2$$
Thus the total volume of this Solid of Revolution is
$$Volume = \int_0^2 \left(\pi(5 - \frac{3}{2}y)^2 - \pi(2)^2\right) dy$$</td>
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<td>4.</td>
<td>Discuss the continuity of $f(x,y) = \begin{cases} \dfrac{3xy^2}{x^2+y^4}; (x,y) \neq (0,0) \\ 0 \quad ; (x,y) = (0,0) \end{cases}$ at all points of the domain of $f(x,y)$.</td>
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	Soln:– Clearly, $\frac{3xy^2}{x^2+y^4}$, $(x,y) \neq (0,0)$ is a rational function, so it is continuous at all points of $\mathbb{R}^2$ except $(0,0)$ i.e. $D = \{(x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0)\}$ In order for f to be continuous at $(0,0)$, we must show that $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$ Let us take the path $x = my^2$, m is non-zero. We note that $x \to 0$ as $y \to 0$ along the chosen path $x = my^2$ $\therefore \lim_{(x,y)\to(0,0)} \frac{3xy^2}{x^2+y^4}$ $= \lim_{y\to 0} \frac{3\cdot my^2\cdot y^2}{(my^2)^2+y^4}$ $= \lim_{y\to 0} \frac{3m}{m^2+1} = \frac{3m}{m^2+1}$ $= \phi(m)$ (say) We see that the limit depends on the approach path; the limit at $(0,0)$ does not exist, and f is not continuous at $(0,0)$.			
5.	If $u = f(r,s,t)$ and $r = \frac{x}{y}, s = \frac{y}{z}, t = \frac{z}{x}$, then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$. Soln:– $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$ $= \frac{\partial u}{\partial r}\left(\frac{1}{y}\right) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}\left(\frac{-z}{x^2}\right)$ $= \frac{1}{y}\frac{\partial u}{\partial r} - \frac{z}{x^2}\frac{\partial u}{\partial t}$ —— ①	10	2	2

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r}\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial s}\left(\frac{1}{z}\right) + \frac{\partial u}{\partial t}(0)$$

$$= -\frac{x}{y^2}\frac{\partial u}{\partial r} + \frac{1}{z}\frac{\partial u}{\partial s} \quad \text{—— (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}\left(-\frac{y}{z^2}\right) + \frac{\partial u}{\partial t}\left(\frac{1}{x}\right)$$

$$= \frac{-y}{z^2}\frac{\partial u}{\partial s} + \frac{1}{x}\frac{\partial u}{\partial t} \quad \text{—— (3)}$$

$$\therefore \quad x\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial r} - \frac{z}{x}\frac{\partial u}{\partial t}$$

$$y\frac{\partial u}{\partial y} = \frac{-x}{y}\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial s}$$

$$z\frac{\partial u}{\partial z} = \frac{-y}{z}\frac{\partial u}{\partial s} + \frac{z}{x}\frac{\partial u}{\partial t}.$$

$$\therefore \sum x\frac{\partial u}{\partial x} = 0.$$