SLOT: G2+TG2

Vellore Institute of Technology

(Deemed to be University under section 3 of UGC Act, 1956)

SCHOOL OF ADVANCED SCIENCES
DEPARTMENT OF MATHEMATICS
CONTINUOUS ASSESSMENT TEST – I
FALL SEMESTER 2023-24

Programme Name & Branch : **B.Tech.**
Course Code : **BMAT101L**
Course Name : **Calculus**
Exam Duration : **90 minutes** **Maximum Marks: 50**

General instruction(s): Answer All Questions ($5 \times 10 = 50$).

Q. No.	Question	Marks
1.	Verify Lagrange's mean value theorem for the function $f(x) = 2\sin x - \sin 2x$ in the interval $[0, \pi]$. Then find all values of c that satisfy the conclusion of the mean value theorem.	10
2.	If the function is defined by $f(x) = \frac{1}{2}x^2 \ln x$, then (i) find the critical points, local maximum and local minimum, (ii) find the intervals where $f(x)$ is increasing and decreasing, (iii) identify the intervals where $f(x)$ is concave up and concave down, and find the points of inflection, if exist.	10
3.	Let R be the region enclosed by $y = x^3$ and $x = y^4$ then find the volume of the solid generated by revolving the region R about the line $y = -\frac{1}{2}$.	10
4.	Discuss the continuity of the following function at the origin: $$f(x,y) = \begin{cases} \dfrac{x(\cos y - 1)}{x^3 + y^3}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$	10
5.	If $w = \sin^{-1} u$, where $u = \frac{x^2+y^2}{x+y}$, then find $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y}$.	10

FS 2023-24, BMAT101L, Slot G2, Key

① $f(x) = 2\sin x - \sin 2x$ in $[0,\pi]$.

(i) $f(x)$ is continuous on $[0,\pi]$

(ii) $f'(x) = 2\cos x - 2\cos 2x$ exists in $(0,\pi)$.

Hence L.M.V.T's conditions are satisfied.

So, $\exists\ c \in (0,\pi)$ s.t. $f'(c) = \dfrac{f(\pi)-f(0)}{\pi - 0}$

$$2\cos c - 2\cos 2c = \frac{0-0}{\pi}$$

$$\cos c - 2\cos^2 c + 1 = 0$$

$$2\cos^2 c - \cos c - 1 = 0$$

$$2\cos^2 c - 2\cos c + \cos c - 1 = 0$$

$$2\cos c(\cos c - 1) + 1(\cos c - 1) = 0$$

$$(\cos c - 1)(2\cos c + 1) = 0$$

$\cos c - 1 = 0$ or $2\cos c + 1 = 0$

$\cos c = 1$ $\quad$ $\cos c = -\frac{1}{2}$

$c = 2n\pi$ $\quad$ $\boxed{c = \frac{2\pi}{3}} \in (0,\pi)$

i.e., $c = 0, 2\pi \notin (0,\pi)$

② $f(x) = \frac{1}{2}x^2 \ln x$; Domain is $(0,\infty)$.

$f'(x) = x\ln x + \frac{x}{2}$; $f''(x) = \ln x + \frac{3}{2}$

(i) Critical point is $\frac{1}{\sqrt{e}}$,

No local max.

local min at $\frac{1}{\sqrt{e}}$ is $-\frac{1}{4e}$.

(ii) $f'(x) > 0$; in $\left(\frac{1}{\sqrt{e}}, \infty\right)$

$f'(x) < 0$; in $\left(0, \frac{1}{\sqrt{e}}\right)$

(iii) Concave up on $(e^{-3/2}, \infty)$

Concave down on $(0, e^{-3/2})$

Point of inflection at $x = e^{-3/2}$.

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③ $y = x^3$ & $x = y^4$ about the line $y = -\frac{1}{2}$.

$$V = \int_0^1 \pi \left[\left(x^{1/4} + \frac{1}{2}\right)^2 - \left(x^3 + \frac{1}{2}\right)^2 \right] dx$$

$$= \frac{451\pi}{450}.$$

④ $$f(x,y) = \frac{x(\cos y - 1)}{x^3 + y^3}$$

Along the path $y = mx$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{\substack{x\to 0 \\ y = mx}} \frac{x(\cos mx - 1)}{x^3 + m^3x^3}$$

$$= \lim_{x\to 0} \frac{\cos mx - 1}{x^2 + m^3x^2} \quad ; \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{-m \sin mx}{2x(1+m^3)} \quad ; \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{-m^2 \cos mx}{2(1+m^3)}$$

$$= \frac{-m^2}{2(1+m^3)}$$; Limit doesn't exist at origin hence function is discontinuous at origin

⑤ $w = \sin^{-1} u$

or $$\sin w = \frac{x^2 + y^2}{x + y}$$

$$\cos w \cdot \frac{\partial w}{\partial x} = \frac{(x+y)\cdot 2x - (x^2+y^2)}{(x+y)^2} \quad ; \quad \cos w \cdot \frac{\partial w}{\partial y} = \frac{(x+y)2y - (x^2+y^2)}{(x+y)^2}$$

$$\therefore\ x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = \sec w \times \sin w$$

$$= \tan w.$$

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