VIT
Vellore Institute of Technology

SCHOOL OF ADVANCED SCIENCES

Fall Semester 2023-2024

Continuous Assessment Test – II

Programme Name & Branch B.Tech. Computer Science and Engineering

Slot: E1+TE1

Course Name & code: Calculus & BMAT101L

Exam Duration: 90 Min. **Maximum Marks:** 50

Q.No.	Question	Max Marks
1.	Test whether the following functions are functionally dependent, and if so, find the relation between them. $x = ue^v \cos w, \quad y = ue^v \sin w, \quad z = u^2 e^{2v}$.	10
2.	Obtain Taylor's series expansion for the function $f(x,y) = e^x \cos y$ about the point $\left(1, \frac{\pi}{4}\right)$ up to second degree terms. Hence, compute $f\left(2, \frac{3\pi}{4}\right)$ approximately.	10
3.	If the temperature T at any point (x, y, z) in space is $T = 400xyz^2$. Find the highest temperature on the surface of the unit sphere $x^2 + y^2 + z^2 = 1$.	10
4.	(a) Sketch the region of integration for the following double integral, $\int_0^{2a} \int_{x^2/4a}^{3a-x} \left(x^2 + y^2\right) dy\, dx$ (b) Evaluate above integral by change of order of integration.	10
5.	Using polar coordinates, evaluate the following double integral $\int_0^a \int_0^{\sqrt{a^2-y^2}} \left(x^2 + y^2\right) dx\, dy$	10

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CAT-2 Key BMAT101L – Calculus – E1+TE1

① Given set of functions are $x = ue^v \cos w$, $y = ue^v \sin w$, $z = u^2 e^{2v}$.

here, the functions x, y, z are functionally dependent if $J\left(\frac{x,y,z}{u,v,w}\right) = 0$ ~~i.e~~ For that, we have

$\frac{\partial x}{\partial u} = e^v \cos w$, $\frac{\partial x}{\partial v} = ue^v \cos w$, $\frac{\partial x}{\partial w} = -ue^v \sin w$.

$\frac{\partial y}{\partial u} = e^v \sin w$, $\frac{\partial y}{\partial v} = ue^v \sin w$, $\frac{\partial y}{\partial w} = ue^v \cos w$.

$\frac{\partial z}{\partial u} = 2ue^{2v}$, $\frac{\partial z}{\partial v} = 2u^2 v e^{2v}$, $\frac{\partial z}{\partial w} = 0$.

consider $J\left(\frac{x,y,z}{u,v,w}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$

$$= \begin{vmatrix} e^v \cos w & ue^v \cos w & -ue^v \sin w \\ e^v \sin w & ue^v \sin w & ue^v \cos w \\ 2ue^{2v} & 2u^2 v e^{2v} & 0 \end{vmatrix} = 0.$$

So, x, y, z are functionally dependent. Then relation b/w them is $\boxed{x^2 + y^2 = z.}$

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② Given function $f(x,y) = e^x \cos y$ — ① and point $(a, b) = (1, \pi/4)$.

here, Taylor series of $f(x,y)$ about a point (a,b) is.

$f(x,y) = f(a,b) + [(x-a) f_x(a,b) + (y-b) f_y(a,b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a,b)$

		$x=1, y=\pi/4$
$f(x,y)$	$e^x \cos y$	$e/\sqrt{2}$
f_x	$e^x \cos y$	$e/\sqrt{2}$
f_y	$-e^x \sin y$	$-e/\sqrt{2}$
f_{xx}	$e^x \cos y$	$e/\sqrt{2}$
f_{yy}	$-e^x \cos y$	$-e/\sqrt{2}$

putting these values in Taylor's theorem, we get

$$e^x \cos y = \frac{e}{\sqrt{2}} + \left[(x-1)\frac{e}{\sqrt{2}} + (y-\tfrac{\pi}{4})(-\tfrac{e}{\sqrt{2}})\right]$$
$$+\frac{1}{2!}\left[(x-1)^2\frac{e}{\sqrt{2}} + 2(x-1)(y-\tfrac{\pi}{4})(-\tfrac{e}{\sqrt{2}}) + (y-\tfrac{\pi}{4})^2(-\tfrac{e}{\sqrt{2}})\right] + \cdots$$

$$e^x \cos y = \frac{e}{\sqrt{2}}\left[1+(x-1)-(y-\tfrac{\pi}{4}) + \frac{(x-1)^2}{2} - (x-1)(y-\tfrac{\pi}{4}) - (y-\tfrac{\pi}{4})^2 + \cdots\right] \quad (3)$$

This is required Taylor's series expansion upto second degree.

Also, To compute $f(2, 3\pi/4)$, take $x=2, y=3\pi/4$ in Eq (3).

Then, $e^2\cos(3\pi/4)$ value approximation is.

$$= \frac{e}{\sqrt{2}}\left[1+(2-1)-(\tfrac{3\pi}{4}-\tfrac{\pi}{4}) + \left(\frac{2-1}{2}\right)^2 - (2-1)(\tfrac{3\pi}{4}-\tfrac{\pi}{4}) - (\tfrac{3\pi}{4}-\tfrac{\pi}{4})^2\right]$$

$\simeq -5.22$.

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(3) Given that temperature T at any point in a space is $T = 400xyz^2$ (1) and to obtain highest temp. on the unit surface $x^2+y^2+z^2=1$ (2)

Consider, $F(x,y,z) = T(x,y,z) + \lambda\,\phi(x,y,z)$ by Lagranges method of undetermined multipliers.

here $\phi(x,y,z) = x^2+y^2+z^2-1$.

So, $F(x,y,z) = 400xyz^2 + \lambda(x^2+y^2+z^2-1)$ (3)

To obtain Stationary values, we have $F_x = F_y = F_z = 0$.

i.e, for $F_x = 0$, $400yz^2 + \lambda(2x) = 0 \Rightarrow \lambda = \dfrac{-200yz^2}{x}$ (4)

for $F_y = 0$, $400xz^2 + \lambda(2y) = 0 \Rightarrow \lambda = \dfrac{-200xz^2}{y}$ (5)

for $F_z = 0$, $800xyz + \lambda(2z) = 0 \Rightarrow \lambda = -400xy$ (6)

Now, from (4) and (5) we have $y/x = x/y \Rightarrow \boxed{x^2 = y^2}$ (7)

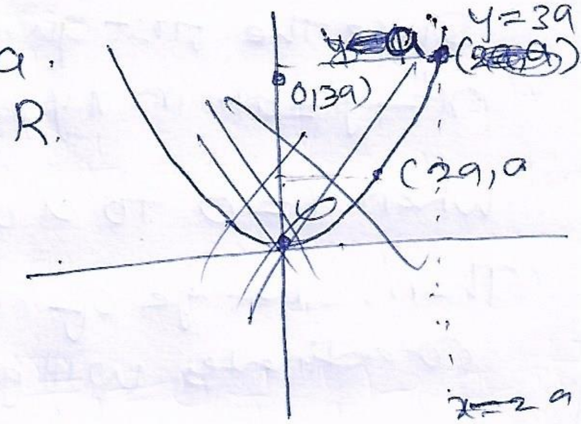
Sly $z = \sqrt{2}x$ ~~then~~ But, $x^2+y^2+z^2=1$ and hence, we have

The maximum temp is $T = 400xyz^2 = 50$

4(a) Given double integral is $\int_0^{2a}\int_{x^2/4a}^{3a-x} (x^2+y^2)\,dy\,dx$

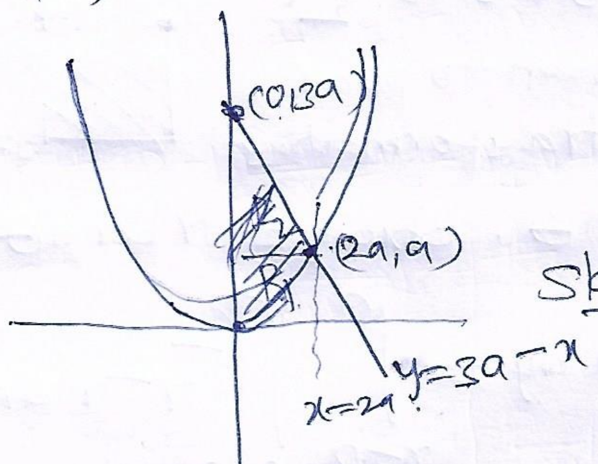
here, region of integration R is bounded by the curves.

$x=0,\ x=2a,\ y=3a-x,\ y=x^2/4a$.



Intersection points of these curves in R.

$(2a,a)$, $(0,3a)$.



Sketch

By change of order of integration region R_1 is divided into R_1 and R_2. i.e, $R = R_1 \cup R_2$.

Where R_1 is enclosed by $y=0$ and $y=a$.
$x=0$ and $x=2\sqrt{ay}$.

Where as R_2 is enclosed by the lines $y=a$ to $y=3a$
$x=0$ to $x=3a-y$.

So, given integral is changed as

$$\iint_R (x^2+y^2)\,dx = \iint_{R_1} (x^2+y^2)\,dx\,dy + \iint_{R_2} (x^2+y^2)\,dx\,dy$$

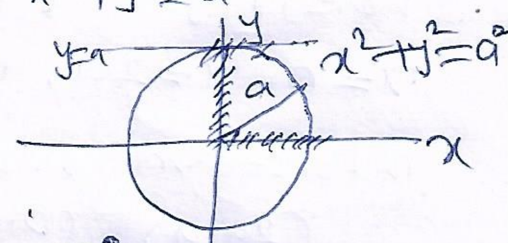
$$= \int_{y=0}^{a}\left[\int_{x=0}^{2\sqrt{ay}} (x^2+y^2)\,dx\right]dy + \int_{y=a}^{3a}\left[\int_{x=0}^{3a-y} (x^2+y^2)\,dx\right]dy.$$

$$= \int_{y=0}^{a}\left[\frac{x^3}{3}+xy^2\right]_0^{2\sqrt{ay}} dy + \int_{y=a}^{3a}\left[\frac{x^3}{3}+xy^2\right]_0^{3a-y} dy.$$

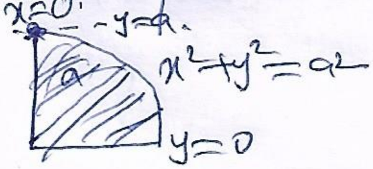
(5) Given double integral is $I = \int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy$ — (1)

here, the region of integration is bounded by the lines $y=0$; $y=a$, $x=0$, $x=\sqrt{a^2-y^2}$, i.e $x^2+y^2=a^2$

Since the first quadrant of the circle $x^2+y^2=a^2$ in xy plane is covered.



when $r=0$ to a and $\theta=0$ to $\pi/2$:



Then, change of cartesian to polar co-ordinates will give transformation

$x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

So, $I = \int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy = \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} r^2 r\,dr\,d\theta$

$= \int_{r=0}^{a} r^3\,dr \cdot \int_{\theta=0}^{\pi/2} d\theta = \left[\frac{r^4}{4}\right]_0^a [\theta]_0^{\pi/2} = \frac{\pi a^4}{8}$