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Vellore Institute of Technology

(Deemed to be University under section 3 of UGC Act, 1956)

SCHOOL OF ADVANCED SCIENCES

Fall Semester 2023-2024

Continuous Assessment Test – II

Programme Name & Branch : B.Tech., Common to all Branches

Slot: G1+TG1

Course Name & code: Calculus and BMAT101L

Exam Duration: 90 Min. **Maximum Marks: 50**

General instruction(s):

Q.No.	Question	Max Marks	CO	BL
1.	Show that $u = x^3 - y^3z^3, v = x^2 + y^2z^2 + xyz,$ $w = x - yz$ are functionally dependent and also find the relation.	10	CO2	2
2.	Expand the function $f(x,y) = \sin(xy)$ in powers of $x - 2$ and $y - \frac{\pi}{4}$ upto third degree.	10	CO2	4
3.	Examine the function $f(x,y) = x^3 + 3xy^2 - 15x^2 + 72x - 15y^2$ for extreme values and saddle points.	10	CO2	5
4.	Evaluate $\int_0^4 \int_{\frac{x^2}{4}}^{8-x} xydydx$ by changing the order of integration.	10	CO3	2
5.	Evaluate $\int_0^3 \int_0^{\sqrt{9-y^2}} y\log(\sqrt{x^2+y^2})\ dxdy$ by change of polar coordinates.	10	CO3	5

Moderator 1:

Moderator 2:

(Estd. u/s 3 of UGC Act 1956)

CAT II – G1 Slot key

(1) Show that $u = x^3 - y^3 z^3$, $v = x^2 + y^2 z^2 + xyz$ and $w = x - yz$ are functionally dependent and also find the relation.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 3x^2 & -3y^2z^3 & -3y^3z^2 \\ 2x+yz & 2yz^2+xz & 2y^2z+xy \\ 1 & -z & -y \end{vmatrix}$$

$$= zy \begin{vmatrix} 3x^2 & -3y^2z^2 & -3y^2z^2 \\ 2x+yz & 2yz+x & 2yz+x \\ 1 & -1 & -1 \end{vmatrix}$$

Take z from C_2 & y from C_3

$= 0$ as C_2 & C_3 are identical.

& $(x^3 - y^3z^3) = (x - yz)(x^2 + y^2z^2 + xyz)$

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2) $f(x,y) = \sin(xy)$, $x = 2$ $y = \pi/4$

$$f(2, \pi/4) = \sin(2\pi/4) = \sin(\pi/2) = 1$$

$$f_x = y\cos(xy) \quad f_x(2,\pi/4) = \pi/4 \cos(\pi/2) = 0$$

$$f_y = x\cos(xy) \quad f_y(2,\pi/4) = 0$$

$$f_{xx} = -y^2\sin(xy) \quad f_{xx}(2,\pi/4) = \frac{-\pi^2}{16}$$

$$f_{xy} = -xy\sin(xy) \quad f_{xy}(2,\pi/4) = \frac{-\pi}{2}$$

$$f_{yy} = -x^2\sin(xy) \quad f_{yy}(2,\pi/4) = -4$$

$$f_{xxx} = -y^3\cos(x,y) \quad f_{xxx} = 0$$

$$f_{xxy} = -2y\sin(xy) - y^2 x\cos(xy) \quad f_{xxy} = -\pi/2$$

$$f_{yyy} = -x^3\cos(xy) \quad f_{yyy} = 0$$

$$f_{xyy} = -2x\sin(xy) - x^2 y\cos(xy) \quad f_{xyy} = -2$$

$$1 + (x-2)^2\frac{\pi^2}{16} - 2(x-2)(y-\pi/4)\frac{\pi}{2} + (y-\pi/4)^2(-4)$$

$$+ \frac{(x-2)^2(y-\pi/4)(-\pi/2)}{3!} + \frac{(x-2)(y-\pi/4)^2(-2)}{3!}$$

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3) Examine the function $f(x,y) = x^3 + 3xy^2 - 15x^2 + 72x - 15y^2$ for extreme values.

$$f(x,y) = x^3 + 3xy^2 - 15x^2 + 72x - 15y^2$$

$$f_x = 3x^2 + 3y^2 - 30x + 72$$

$$f_y = 6xy - 30y$$

$$f_{xx} = 6x - 30$$

$$f_{xy} = 6y$$

$$f_{yy} = 6\cancel{x} - 30$$

Stationary pts

$$f_x = 0, \quad f_y = 0.$$

$$(6x - 30)y = 0$$

$$y = 0 \text{ or } 6x = 30$$

$$y = 0 \text{ or } x = 5.$$

$$f_x = 0 \Rightarrow 3x^2 + 3y^2 - 30x + 72 = 0$$

If $y = 0 \Rightarrow$
$$3x^2 - 30x + 72 = 0$$
$$x^2 - 10x + 24 = 0$$
$$(x-6)(x-4) = 0.$$
$$x = 6 \text{ or } 4.$$

If $x = 5 \Rightarrow$
$$3(25) + 3y^2 - 30(5) + 72 = 0$$
$$75 - 3y^2 - 150 + 72 = 0$$
$$147 - 150 - 3y^2 = 0$$
$$-3 - 3y^2 = 0 \Rightarrow y^2 = 1.$$

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$y = \pm 1.$

Thus $y=0 \Rightarrow x = 6$ or 4

$x = 5 \Rightarrow y = 1$ or -1

stationary pts are

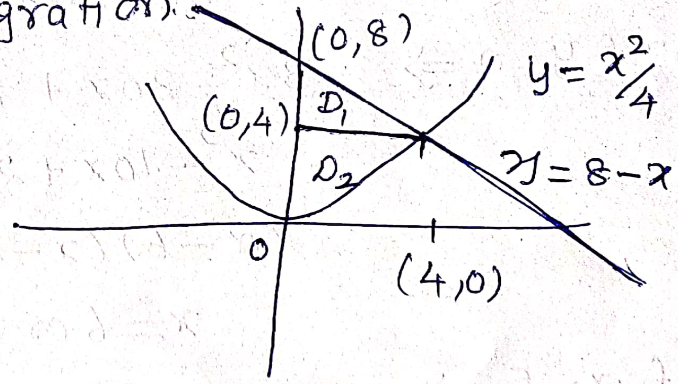
$(6,0), (4,0), (5,1), (5,-1)$

st. pts	r	$rt-s^2$	classification.
$(5,1)$	0	-36	saddle pt
$(5,-1)$	0	-36	saddle pt
$(4,0)$	-6	36	max pt
$(6,0)$	6	36	min pt

$f(4,0) = 112$

$f(6,0) = 108.$

4) Evaluate $\int_0^4 \int_{x^2/4}^{8-x} xy\,dy\,dx$ by changing the order of integration.



$y = x^2/4, \quad y = 8-x.$

$\frac{x^2}{4} = 8-x$

$x^2 = 32-4x$

$x^2+4x-32=0$

$(x+8)(x-4)=0$

$x=4$

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By changing the order of integration

In the region D_1, $x = 0$ to $2\sqrt{y}$, $y = 0$ to 4

In the region D_2, $x = 0$ to $8-y$, $y = 4$ to 8

$$\int_0^4 \int_{x^2/4}^{8-x} xy \, dy \, dx = \int_0^4 \int_0^{2\sqrt{y}} xy \, dx \, dy + \int_4^8 \int_0^{8-y} xy \, dx \, dy.$$

$$= \int_0^4 y \left.\frac{x^2}{2}\right)_0^{2\sqrt{y}} dy + \int_4^8 y \left.\frac{x^2}{2}\right)_0^{8-y} dy$$

$$= \int_0^4 y \left(\frac{4y}{2}\right) dy + \int_4^8 \frac{y(8-y)^2}{2} dy$$

$$= \left.\frac{2y^3}{3}\right)_0^4 + \frac{1}{2} \int_4^8 (64y - 16y^2 + y^3) \, dy$$

$$= \frac{2}{3} 4^3 + \frac{1}{2} \left(\frac{64y^2}{2} - \frac{16y^3}{3} + \frac{y^4}{4} \right)_4^8$$

$$= \frac{2}{3} 4^3 + \frac{160}{3} = \frac{3 \times 4^3}{8} = 24.$$

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(5) $$\int_0^3 \int_0^{\sqrt{9-y^2}} y \log\left(\sqrt{x^2+y^2}\right) dx\, dy$$

$x = r\cos\theta, \quad y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta, \quad 0 \le \theta \le \pi/2, \quad 0 < r < 3$

$$\int_0^3 \int_0^{\sqrt{9-y^2}} y \log\left(\sqrt{x^2+y^2}\right) dx\, dy$$

$$= \int_0^3 \int_0^{\pi/2} r\sin\theta \log\left(\sqrt{r^2}\right) r\, dr\, d\theta$$

$$= \int_0^3 \int_0^{\pi/2} r^2 \log(r)\, dr\, \sin\theta\, d\theta.$$

$$= 9\log 3 - 3 = 6.8875.$$

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