Winter Semester 2023-24
Department of Mathematics
School of Advanced Sciences
Continuous Assessment Test II
KEY

Course Code: BMAT102L **Course Title:** Differential Equations & Transforms
Slot: A2+TA2 **Max. Marks: 50**

Answer all the questions. Each question carries 10 *marks.*

1. **CO 2** **BL2**

(a) Find the Laplace transform of the periodic signal with period $\tau = 2\pi$, given by $f(t) = \begin{cases} 1/2, & 0 < t \leq \pi, \\ 2/3, & \pi < t \leq 2\pi. \end{cases}$ (5 marks)

Solution. We have

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} f(t)\, e^{-st}\, \mathrm{d}t \\
&= \frac{1}{1-e^{-2\pi s}} \left(\frac{1}{2} \int_0^{\pi} e^{-st}\, \mathrm{d}t + \frac{2}{3} \int_{\pi}^{2\pi} e^{-st}\, \mathrm{d}t \right) \\
&= \frac{1}{1-e^{-2\pi s}} \left(-\frac{1}{2} \cdot \frac{e^{-st}}{s} \bigg|_{t=0}^{2\pi} - \frac{2}{3} \cdot \frac{e^{-st}}{s} \bigg|_{t=0}^{2\pi} \right) \\
&= \frac{1}{1-e^{-2\pi s}} \left(\frac{1}{2s} - \frac{1}{3s} e^{-\pi s} - \frac{2}{3s} e^{-2\pi s} \right).
\end{aligned}$$

(b) Find $\mathscr{L}\left\{ t\left(e^{-at} + e^{at}\right)^2 \right\}$. (5 marks)

Solution.

$$f(t) = t\left(e^{-at} + e^{at}\right)^2 = t\left(e^{-2at} + e^{2at} + 2e^{-at} \cdot e^{at}\right) = t\left(e^{-2at} + e^{2at} + 2\right)$$

Since $\mathscr{L}\{t\} = 1/s^2$, by the first shifting property,

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \mathscr{L}\left\{ t\left(e^{-2at} + e^{2at} + 2\right) \right\} = \mathscr{L}\left\{ te^{-2at} \right\} + \mathscr{L}\left\{ te^{2at} \right\} + \mathscr{L}\{t\} \\
&= \frac{1}{(s+2a)^2} + \frac{1}{(s-2a)^2} + \frac{1}{s^2}.
\end{aligned}$$

2. **CO 2** **BL2**

Find the convolution of $f(t) = t^2$ and $g(t) = (\sin 5t)/5$. Use this information to obtain the inverse Laplace transform of $P(s) = 1/s^3(s^2 + 25)$

Solution.

$$\begin{aligned}
f(t) * g(t) &= \int_0^t f(u)g(t-u)\,\mathrm{d}u = \frac{1}{5}\int_0^t u^2 \sin(5t-5u)\,\mathrm{d}u \\
&= \frac{1}{5}\left|\frac{1}{5}u^2\cos(5t-5u) + \frac{1}{25}2u\sin(5t-5u) - \frac{2}{125}\cos(5t-5u)\right|_{u=0}^{t} \\
&= \frac{1}{5}t^2 + \frac{2}{625}\cos 5t - \frac{2}{625}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathscr{L}^{-1}\{P(s)\} &= \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{2}{s^3}\right\} * \mathscr{L}^{-1}\left\{\frac{1}{s^2+25}\right\} = \frac{1}{2}f(t) * g(t) \\
&= \frac{1}{2}\left(\frac{1}{5}t^2 + \frac{2}{625}\cos 5t - \frac{2}{625}\right) = \frac{1}{10}t^2 + \frac{1}{625}\cos 5t - \frac{1}{625}.
\end{aligned}$$

3. **CO 4** **BL3**

Solve the initial value problem $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \frac{\mathrm{d}x}{\mathrm{d}t} - 12x = H(t-3)$ with $x(0) = -1$, $x'(0) = -4$ through the Laplace transform, where $H(t-a)$ denotes the Heaviside's unit step function with $a \geq 0$.

Solution. Applying the Laplace transform to the ode, then the initial conditions, and simplifying

$$(s^2 - s - 12)F(s) = -(s+3) + \frac{1}{s}e^{-3s}$$

or

$$\begin{aligned}
F(s) &= -\frac{1}{s-4} + \frac{1}{s(s+3)(s-4)}e^{-3s} \\
&= -\frac{1}{s-4} + e^{-3s}\left(-\frac{1}{12}\cdot\frac{1}{s} + \frac{1}{21}\cdot\frac{1}{s+3} + \frac{1}{28}\cdot\frac{1}{s-4}\right)
\end{aligned}$$

Using the second shifting property,

$$x(t) = \mathscr{L}^{-1}\{F(s)\} = e^{-4t} + \left(-\frac{1}{12} + \frac{1}{21}\cdot e^{-3(t-3)} + \frac{1}{28}\cdot e^{4(t-3)}\right)H(t-3).$$

4. **CO 3** **BL2**

Find the Fourier series of the half-wave rectified spatial signal

$$f(x) = \begin{cases} 0, & -\pi < x \leq 0, \\ \frac{x}{3}, & 0 < x < \pi. \end{cases}$$

Use it to compute the sum of the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2}\cdots$. What about the sum of the Fourier series at $\pm\pi$?

Solution. The Fourier coefficients are

$$a_0 = \frac{\pi}{6},\ a_n = \frac{1}{3\pi}\frac{(-1)^n - 1}{n^2},\ b_n = \frac{1}{3}\frac{(-1)^n}{n},\ n \geq 1.$$

The Fourier series is

$$f(x) = \frac{\pi}{12} + \frac{1}{3\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos nx + \frac{1}{3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx, -\pi < x < \pi.$$

Writing $x = 0$ in the Fourier series,

$$\frac{\pi}{12} + \frac{1}{3\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2} = \frac{1}{2}(f(0-0) + f(0+0)) = 0 \text{ or } \sum_{n \text{ is odd}}\frac{1}{n^2} = \frac{\pi^2}{8}.$$

The sum of the Fourier series at $x = \pi$ equals $\frac{1}{2}(f(\pi - 0) + f(\pi + 0)) = \pi/12$, and the sum of the Fourier series at $x = -\pi$ is equal $\frac{1}{2}(f(-\pi - 0) + f(-\pi + 0)) = \pi/12$.

5. **CO 3** **BL2**

(a) Find the half-range sine series of $f(x) = e^{-x}$ in $(0, l)$.

Solution. The Fourier coefficients are

$$\begin{aligned} b_n &= \frac{2}{l}\int_0^l f(x)\sin(n\pi x/l)\ \mathrm{d}x = \frac{2}{l}\int_0^l e^{-x}\sin(n\pi x/l)\ \mathrm{d}x \\ &= \frac{2}{l}\left|\frac{e^{-x}}{1 + (n^2\pi^2/l^2)}\left(-\sin(n\pi x/l) - \frac{n\pi}{l}\cos(n\pi x/l)\right)\right|_{x=0}^{l} \\ &= \frac{2l}{l^2 + \pi^2 n^2}\left(e^{-l}\sin n\pi - \frac{n\pi}{l}\cos n\pi + 0 + \frac{n\pi}{l}\cos 0\right) \\ &= \frac{2n\pi}{l^2 + \pi^2 n^2}(1 - (-1)^n) \text{ for } n \geq 1. \end{aligned}$$

Therefore,

$$e^{-x} = \sum_{n=1}^{\infty}\frac{2n\pi}{l^2 + \pi^2 n^2}(1 - (-1)^n)\sin\left(\frac{n\pi x}{l}\right), 0 \leq x \leq l.$$

(b) Given that $x^3 - \pi^2 x = \sum_{n=1}^{\infty}\frac{12(-1)^n}{n^3}\sin nx$ for all $0 \leq x \leq \pi$, find the sum of the series $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$.

Solution. Write $x = \pi/2$ in this. Then $\sum_{n=1}^{\infty}\frac{12(-1)^n}{n^3}\sin(n\pi/2) = \frac{\pi^3}{8} - \frac{\pi^3}{2} = -\frac{3\pi^3}{8}$ or $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}$, since $\sin n\pi = 0$ and $\sin(n\pi/2) = \pm 1$ whenever $n \equiv 1 (\text{mod } 4)$ or $n \equiv 3 (\text{mod } 4)$.