VIT

Vellore Institute of Technology

(Deemed to be University under section 3 of UGC Act, 1956)

SCHOOL OF ADVANCED SCIENCES

Winter Semester 2023-2024

Continuous Assessment Test –II

Programme Name & Branch : **B.Tech.,**

Slot : B2+TB2+TBB2

Course Name & code : Differential Equations and Transforms & BMAT102L

Exam Duration: 90 Min. **Maximum Marks: 50**

General instruction(s): Answer ALL Questions

Q.No.	Question	Max Marks
1.	(i) Evaluate the Laplace transform of the function $\frac{\sin 3t \cos 2t}{t}$. (ii) Let $f(t)$ be periodic with period 4, where $f(t) = \begin{cases} 3t, & 0 < t < 2 \\ 6, & 2 < t < 4 \end{cases}$. Find the Laplace transform of the function. (5M+5M)	10 CO2
2.	Find the inverse Laplace transform of $\frac{s}{s^4+s^2+1}$ using the method of partial fraction.	10 CO2
3.	An inductor of 2 henrys, a resistor of 16 ohms and a capacitor of 0.02 farads are connected in series with an e.m.f of $H(t-3)$ volts. At t=0, the charge on the capacitor and current in circuit are zero. Using the Laplace transform, find the charge at any time t>0.	10 CO2
4.	Obtain the Fourier series for the function $f(x) = \begin{cases} x, & in\ 0 < x < 1 \\ 1-x, & in\ 1 < x < 2 \end{cases}$ and hence find the value of $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$.	10 CO3
5.	Find the half range Fourier sine series expansion of $x(\pi - x)$ in the interval $(0, \pi)$. Also evaluate the root mean square value of the function.	10 CO3

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B2+TB2+TBB2 BMAT102L

CAT-II Answer key.

① (i) $\frac{1}{2}\left[L\left\{\frac{\sin 5t}{t}\right\} + L\left\{\frac{\sin t}{t}\right\}\right]$

$= \frac{1}{2}\left[\int_s^\infty \frac{5}{s^2+25}\,ds + \int_s^\infty \frac{1}{s^2+1}\cdot ds\right]$

$= \frac{1}{2}\left[\tan^{-1}(s/5)\Big|_s^\infty + \tan^{-1}(s)\Big|_s^\infty\right]$

$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}(s/5) + \frac{\pi}{2} - \tan^{-1}s\right]$

$= \frac{1}{2}\left[\pi - \tan^{-1}(s/5) - \tan^{-1}(s)\right]$

(ii) $L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}\cdot f(t)\,dt.$

Here $T=4$.

$= \frac{1}{1-e^{-4s}}\left[\int_0^2 3t\cdot e^{-st}\cdot dt + \int_2^4 6e^{-st}\cdot dt\right]$

$u=t \quad dv = e^{-st}\cdot dt$

$du=dt \quad v = \frac{e^{-st}}{-s}$

$= \frac{1}{1-e^{-4s}}\left[\frac{-3}{s}(2e^{-2s}) - \frac{3}{s^2}(e^{-st})_0^2 - \frac{6}{s}\left[e^{-4s} - e^{-2s}\right]\right]$

$= \frac{1}{1-e^{-4s}}\left[\frac{-3}{s^2}e^{-2s} + \frac{3}{s^2} - \frac{6}{s}e^{-4s}\right]$ //

(2) $$\frac{s}{(s^2-s+1)(s^2+s+1)} = \frac{As+B}{s^2-s+1} + \frac{Cs+D}{s^2+s+1}$$ (2)

$\Rightarrow$ $A+C=0$; $A+B-C+D=0$; $A+B+C-D=1$;
$B+D=0$.

on solving, $A=C=0$, $B=\frac{1}{2}$; $D=-\frac{1}{2}$.

$$\therefore L^{-1}\left\{\frac{s}{s^4+s^2+1}\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{s^2-s+1}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{s^2-s+1}\right\}$$

$$\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2 \qquad \left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2.$$

$$= \frac{1}{2}\cdot e^{-\frac{1}{2}t}\cdot\frac{2}{\sqrt{3}}\cdot \sin\left(\frac{\sqrt{3}}{2}\right)t - \frac{1}{2}\cdot e^{\frac{1}{2}t}\cdot\frac{2}{\sqrt{3}}\cdot\sin\left(\frac{\sqrt{3}t}{2}\right)$$

(3) Given $L=2$; $R=16$; $C=0.02$. $E(t)=H(t-3)$.

$$L\cdot\frac{di}{dt}+Ri+\frac{q}{C}=E(t)$$

$$\Rightarrow 2\frac{d^2q}{dt^2}+16\frac{dq}{dt}+50q=H(t-3)$$

$$2\left[s^2L(s)-s\underbrace{q(0)}_{0}-\underbrace{q'(0)}_{0}\right]+16\left[sL(s)-\underbrace{q(0)}_{0}\right]+50L(s) = L\{H(t-3)\}$$

$$2(s^2+8s+25)\cdot L(s)=\frac{e^{-3s}}{s}.$$

$$L(s)=\frac{1}{2}\cdot e^{-3s}\cdot\frac{1}{(s^2+8s+25)\cdot s}$$

$$\Rightarrow q(t)=\frac{1}{2}\cdot L^{-1}\left\{e^{-3s}\cdot\frac{1}{s(s^2+8s+25)}\right\}.$$

using partial fraction.

$$\frac{1}{s(s^2+8s+25)} = \frac{A}{s} + \frac{Bs+C}{s^2+8s+25}.$$

$$\Rightarrow A = \frac{1}{25};\ B = \frac{-1}{25};\ C = \frac{-8}{25}.$$

$$\therefore q(t) = \frac{1}{2\times 25} L^{-1}\left\{e^{-3s}\cdot\frac{1}{s}\right\} - \frac{1}{2\times 25} L^{-1}\left\{e^{-3s}\cdot\frac{s}{s^2+8s+25}\right\}$$

$$-\frac{1}{2}\times\frac{8}{25}\cdot L^{-1}\left\{e^{-3s}\cdot\frac{1}{\underbrace{s^2+8s+25}_{(s+4)^2+3^2}}\right\}$$

Using

$$L^{-1}\left\{e^{-as}\cdot L(s)\right\} = f(t-a)\cdot H(t-a)$$

~~Here~~ $a=3$.

I - term $\rightarrow$ $\frac{1}{50}\cdot f(t-3)\cdot H(t-3)$, where $f(t)=1$.

II - term $\rightarrow$ $\frac{-1}{50}\cdot f(t-3)\cdot H(t-3)$, where $f(t) = e^{-4t}\cdot\cos 3t$.

III - term $\rightarrow$ $-\frac{8}{50}\, f(t-3)\cdot H(t-3)$

where $f(t) = \frac{1}{3} e^{-4t}\cdot\sin 3t$.

(4). $l=1$, $a_0 = \frac{1}{1}\int_0^2 f(x)\,dx$; $a_n = \frac{1}{1}\int_0^2 f(x)\cos n\pi x\,dx$

$$b_n = \frac{1}{1}\int_0^2 f(x)\cdot\sin n\pi x\,dx.$$

$$a_0 = \int_0^1 x\,dx + \int_1^2 (1-x)\,dx = 0$$

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$$a_n = \int_0^1 x\cdot\cos n\pi x\,dx + \int_1^2 (1-x)\cos n\pi x\,dx.$$

$$= \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right] = \begin{cases} \frac{-4}{n^2\pi^2}, & n\text{-odd} \\ 0, & n\text{-even} \end{cases}$$

$$b_n = \int_0^1 x\cdot\sin n\pi x\,dx + \int_1^2 (1-x)\sin n\pi x\,dx.$$

$$= \frac{1-(-1)^n}{n\pi} = \begin{cases} \frac{2}{n\pi}, & n\text{-odd} \\ 0, & n\text{-even} \end{cases}$$

$$\therefore f(x) = \sum_{n=1,3,\dots}^{\infty} \frac{-4}{n^2\pi^2}\cdot\cos n\pi x + \sum_{n=1,3,\dots}^{\infty} \frac{2}{n\pi}\cdot\sin n\pi x.$$

required Fourier series.

Put $x=1$, $f(1) = \frac{1}{2}\left[f(1-) + f(1+)\right] = \frac{1}{2}$

$$\therefore \frac{1}{1^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{8}.$$

5 $l = \pi$. $f(x) = \sum_{n=1}^{\infty} b_n\cdot\sin nx$. $\rightarrow$ is the required HRF sine series

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\sin nx\,dx = \frac{4}{n^3\pi}\left[1-(-1)^n\right].$$

$$\left[f(x)\right]_{rms} = \sqrt{\frac{\int_a^b [f(x)]^2\,dx}{b-a}}, \text{ where } a=0,\ b=\pi$$

$$\left(f(x)\right)^2 = x^2(\pi^2 + x^2 - 2\pi x)$$

$$\int_0^{\pi} [f(x)]^2\,dx = \frac{\pi^5}{30} \qquad \therefore \left(f(x)\right)_{rms} = \sqrt{\frac{\pi^4}{30}} = \frac{\pi^2}{\sqrt{30}}$$