School of Advanced Sciences
Department of Mathematics
WINTER SEMESTER 2023-2024
CONTINUOUS ASSESSMENT TEST – II

Course Code: BMAT102L
Course Name: Differential Equations and Transforms **Slot:** C1+TC1+TCC1
Duration: 90 Minutes **Answer all the Questions** **Max. Marks: 50**

General instruction(s):

Students are permitted to bring any number of text books and hand written note books (class notes)

Q. No	Question	Marks	Course Outcome (CO)	Bloom's Taxonomy (BL)
1.	(a) Find the Laplace transform of the periodic function $f(t)$ of period $\frac{2\pi}{n}$, Where $f(t) = \begin{cases} 2, & in\ 0 < t < \frac{1}{2} \\ 0, & in\ \frac{1}{2} \leq t < \frac{2\pi}{n} \end{cases}$ (b) Evaluate $\int_0^\infty \frac{e^{-3t} - e^{-6t}}{t}\, dt$, Using the Laplace transform.	5+5	CO2	BL2
2.	Using the convolution theorem, evaluate $L^{-1}\left\{\frac{s^2}{(s^2-a^2)(s^2+a^2)}\right\}$	10	CO2	BL2
3.	Solve the differential equation, $$2\frac{d^2y}{dx^2} + 10y = 3H(t-12) - 5\delta(t-4),$$ $$y(0) = -1, \quad y'(0) = -2.$$	10	CO4	BL3
4.	Obtain the Fourier series of period 2π given by $f(x) = \begin{cases} -\pi x - x^2 & -\pi < x < 0 \\ \pi x - x^2 & 0 < x < \pi \end{cases}$ and Deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$	10	CO3	BL3
5.	Find the half-range cosine series of the function $f(x) = 6x^2 - 6x + 1$, in $(0,1)$ and Deduce the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$	10	CO3	BL2

WS (23-24) - BMAT102L - CAT2 - C1+TC1+TCC1 - KEY

① ⓐ $L\{f(t)\} = \frac{1}{1-e^{-\frac{2\pi s}{n}}} \int_0^{\frac{2\pi}{n}} e^{-st} f(t)\,dt$

$= \frac{1}{1-e^{-\frac{2\pi s}{n}}} \left\{ \int_0^{1/2} e^{-st}.2\,dt + \int_{1/2}^{\frac{2\pi}{n}} e^{-st}.0\,dt \right\}$

$= \frac{2(1-e^{-s/2})}{s(1-e^{-\frac{2\pi s}{n}})}.$

we have

ⓑ $\int_0^\infty e^{-st}.\left(\frac{e^{-3t}-e^{-6t}}{t}\right)dt = L\left\{\frac{e^{-3t}-e^{-6t}}{t}\right\} \longrightarrow (*)$

Now, $L\left\{\frac{e^{-3t}-e^{-6t}}{t}\right\} = \int_s^\infty \left(\frac{1}{s+3} - \frac{1}{s+6}\right)ds$

$= \log\left(\frac{s+3}{s+6}\right)\Big]_s^\infty = \log\left(\frac{s+6}{s+3}\right)$

$\therefore$ from $(*)$, we get

$\int_0^\infty e^{-st}\left(\frac{e^{-3t}-e^{-6t}}{t}\right)dt = \log\left(\frac{s+6}{s+3}\right)$

Taking $s=0$, we get $\int_0^\infty \left(\frac{e^{-3t}-e^{-6t}}{t}\right)dt = \log_e 2.$

② $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2-a^2)}\right\} = L^{-1}\left\{\frac{s}{s^2+a^2} \cdot \frac{s}{s^2-a^2}\right\}$

$= \cos at * \cosh at$

$= \int_0^t \cos au \cdot \cosh a(t-u)\,du$

$= \int_0^\infty \cos au \cdot \left[\frac{e^{a(t-u)}+e^{-a(t-u)}}{2}\right]du$

$$= \frac{1}{2}\left\{ e^{at}\int_0^t e^{-au}\cos au\,du + e^{-at}\int_0^t e^{au}\cdot\cos au\,du \right\}$$

$$= \frac{1}{4a}\left[2\sin at + e^{at} - e^{-at}\right] = \frac{1}{2a}\left[\sin at + \frac{e^{at}-e^{-at}}{2}\right]$$

$$\therefore\ L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2-a^2)}\right\} = \frac{1}{2a}\left[\sin at + \sinh at\right]$$

(3) $L\{2y''(t) + 10y(t)\} = 3L\{H(t-12)\} - 5L\{\delta(t-4)\}$

$$\Rightarrow 2\{s^2 L y(t) - s\,y(0) - y'(0)\} + 10 L\{y(t)\} = 3\frac{e^{-12s}}{s} - 5e^{-4s}$$

$\boxed{y(0)=-1,\ y'(0)=-2}$

$$\Rightarrow L\{y(t)\} = \frac{3}{2}\frac{e^{-12s}}{s(s^2+5)} - \frac{5}{2}\frac{e^{-4s}}{s^2+5} - \left(\frac{s+2}{s^2+5}\right)$$

$$\Rightarrow y(t) = \frac{3}{2}L^{-1}\left\{e^{-12s}\cdot\frac{1}{s(s^2+5)}\right\} - \frac{5}{2}L^{-1}\left\{e^{-4s}\cdot\frac{1}{s^2+5}\right\}$$
$$- L^{-1}\left\{\frac{s}{s^2+(\sqrt{5})^2}\right\} - \frac{2}{\sqrt{5}}L^{-1}\left\{\frac{\sqrt{5}}{s^2+(\sqrt{5})^2}\right\}$$

$$\Rightarrow y(t) = \frac{3}{2}L^{-1}\left\{e^{-12s}\left(\frac{1}{5s} - \frac{s}{5(s^2+5)}\right)\right\} - \frac{5}{2\sqrt{5}}\sin\sqrt{5}(t-4)\cdot H(t-4)$$
$$- \cos\sqrt{5}t - \frac{2}{\sqrt{5}}\sin\sqrt{5}t$$

Hence

$$y(t) = \frac{3}{10}\left[1-\cos\sqrt{5}(t-12)\right]H(t-12)$$
$$- \frac{\sqrt{5}}{2}\sin\sqrt{5}(t-4)\cdot H(t-4) - \cos\sqrt{5}t - \frac{2}{\sqrt{5}}\sin\sqrt{5}t.$$

(4) Given function $f(x) = \begin{cases} -\pi x - x^2 & \text{for } -\pi < x < 0 \\ \pi x - x^2 & \text{for } 0 < x < \pi \end{cases}$

is even.

Therefore, the Fourier series for $f(x)$ in the interval $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right).$$

Here $l = \pi$ and $a_0 = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\,dx = \frac{\pi^2}{3}$,

$$a_n = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\cos(nx)\,dx = \frac{-2}{n^2}\left(1+(-1)^n\right).$$

Hence, $f(x) = \frac{\pi^2}{6} - 2\sum_{n=1}^{\infty} \frac{[1+(-1)^n]}{n^2}\cos(nx)$

Taking $x=0$, we get

$$\frac{f(0+)+f(0-)}{2} = \frac{\pi^2}{6} - 2\sum_{n=2,4,6,\ldots} \frac{1}{n^2}$$

or, $-2\left\{\frac{2}{2^2} + \frac{2}{4^2} + \frac{2}{6^2} + \cdots\right\} = 0 - \frac{\pi^2}{6}$

$\Rightarrow \quad \frac{2}{4}\left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right\} = \frac{\pi^2}{12}$

Thus, $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$

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(5) The half-range cosine series for $f(x) = 6x^2 - 6x + 1$ in the interval $(0, l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{l}\right).$$

Here $l = 1$, $a_0 = \frac{2}{l}\int_0^l f(x)\,dx = 2\int_0^1 (6x^2 - 6x + 1)\,dx = 0$

and $a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

$$= 2\int_0^1 (6x^2 - 6x + 1)\cos(n\pi x)\,dx$$

$$= \frac{12}{n^2\pi^2}\left(1 + (-1)^n\right)$$

Therefore, $f(x) = \frac{12}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1 + (-1)^n}{n^2}\right)\cos(n\pi x)$

Taking $x = 0$, we get

$$\frac{12}{\pi^2}\left[\sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\right] = 1$$

$$\Rightarrow \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \frac{\pi^2}{12}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \frac{\pi^2}{12} - \frac{\pi^2}{6}$$

Hence, $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}$