VIT

Vellore Institute of Technology

School of Advanced Sciences

Fall Semester 2024-2025

Continuous Assessment Test – I

Programme Name & Branch: B.Tech

Slot: B2+TB2+TBB2

Course Name & code: Complex Variables and Linear Algebra & BMAT201L

Class Number (s): Common to all B2+TB2+TBB2 slot

Exam Duration: 90 Min. **Maximum Marks: 50**

General instruction(s):

Q.No.	Question	Max Marks
1.	Show that the function $f(z)=\begin{cases}\dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2}, & if \quad z\neq 0\\ 0, & if \quad z=0\end{cases}$ is not analytic at origin although Cauchy-Riemann equations are satisfied at the origin.	10
2.	In the two dimensional fluid flow, the stream function is given by $\psi(x,y)=\dfrac{\sin 2x}{\cosh 2y-\cos 2x}$. Find the complex potential function $f(z)=\phi+i\psi$ and hence find the velocity potential function $\phi(x,\ y)$. $v^2=16(4-u)$	10
3.	Find the image of the region bounded by the lines $x=2,\ x=3,\ y=2,\ y=3$ under the transformation $w=z^2$.	10
4.	Determine the bilinear transformation that maps the points $1,\ i,\ -1$ in z-plane onto the points $i,\ 0,\ -i$ respectively in w-plane and hence find its invariant points. $w=\frac{1+iz}{1-iz}$ $-\frac{1}{2}(1+i\pm\sqrt{6i})$	10
5.	Expand $f(z)=\dfrac{1}{(z-1)(z-2)}$ as a Laurent's series in the region (i) $1<\lvert z\rvert<2$ (ii) $0<\lvert z-1\rvert<1$.	10

$\frac{1}{z-2}-\frac{1}{z-1}$

$-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$

$-\frac{1}{z-1}+\frac{1}{(z-1)-1}$

$-[1-(z-1)]^{-1}$

$-\frac{1}{z-1}$

CAT-I

BMAA2012 - Key Fall-2024-2025

(1) Given $f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

<u>verifying C-R equations:</u>

$$\left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = 1$$

$$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = -1$$

$$\left(\frac{\partial v}{\partial x}\right)_{(0,0)} = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x} = 1, \quad \left(\frac{\partial v}{\partial y}\right)_{(0,0)} = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y} = 1$$

<u>To find $f'(0)$:-</u>

$$\lim_{\substack{x\to 0 \\ y\to 0}} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = 0$$

$$\lim_{\substack{y\to 0 \\ x\to 0}} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = 0$$

<u>Along the path $y = mx$</u>

$$f'(0) = \lim_{\substack{y\to mx \\ x\to 0}} \frac{f(z) - f(0)}{z} = \lim_{x\to 0} \frac{x^3(1+i) - m^3x^3(1-i)}{(x^2+y^2)(x+iy)}$$

$$= \frac{1-m^3 + i(1+m^3)}{(1+m^2)(1+im)}$$ which assumes different values as m varies.

So, $f'(z)$ does not exist.

(2) Given $\psi = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

Let $f(z) = \phi(x,y) + i\psi(x,y)$ be the complex potential function.

we know that $f'(z) = \dfrac{\partial \phi}{\partial x} + i\dfrac{\partial \psi}{\partial x}$

$$f'(z) = \frac{\partial \psi}{\partial y} + i\frac{\partial \psi}{\partial x}$$

$$= \frac{\sin 2x(-2\sinh 2y)}{(\cosh 2y - \cos 2x)^2} + i\,\frac{(\cosh 2y - \cos 2x)\,2\cos 2x - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{-2\sin 2x \cdot \sinh 2y}{(\cosh 2y - \cos 2x)^2} + i\,\frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - 2x)^2}$$

~~By Milne's method~~

By Milne-Thomson's method, we express $f'(z)$ in terms of z by putting $x = z$ and $y = 0$.

$$\therefore f'(z) = \frac{2\cos 2z - 2}{(1-\cos 2z)^2} = -\operatorname{cosec}^2 z$$

$$\Rightarrow f(z) = \cot z + ic.$$

(3) Given $w = z^2$

$$u + iv = (x+iy)^2$$

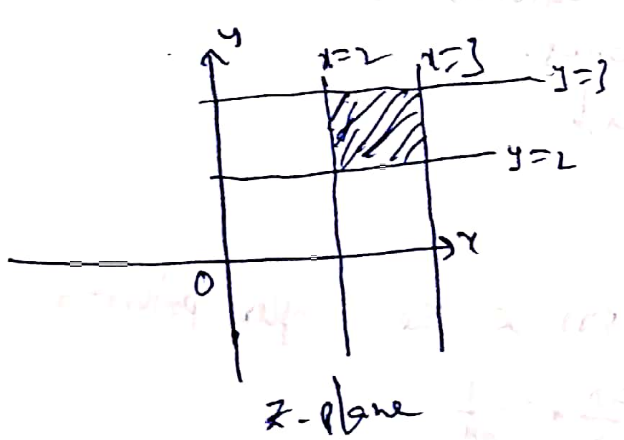
Here $u = x^2 - y^2$, $v = 2xy$

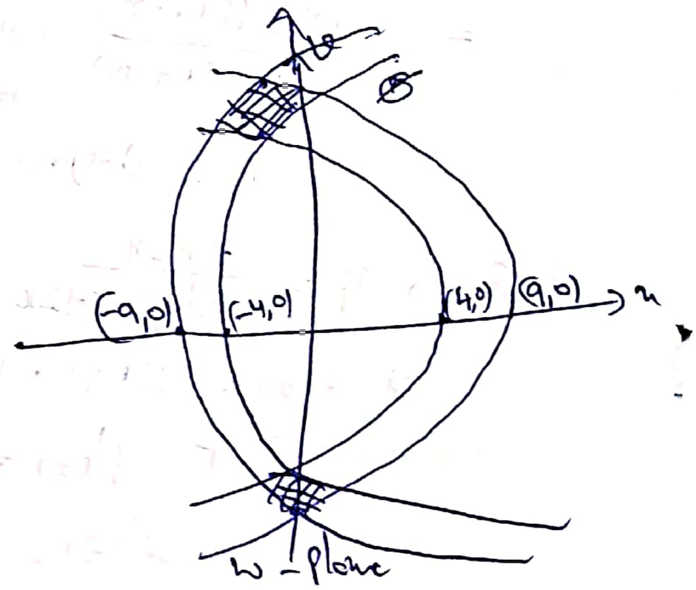
Now, $x = 2 \Rightarrow v^2 = 16(4-u)$ which is a parabola whose vertex is $(4,0)$

$x = 3 \Rightarrow v^2 = 36(9-u)$ which is parabola whose vertex is $(9,0)$

$y = 2 \Rightarrow v^2 = 16(4+u)$ which is a parabola whose vertex is $(-4,0)$

$y = 3 \Rightarrow v^2 = 36(9+u)$ " " " " $(-9,0)$





(4) (a) $z_1=1,\ z_2=i,\ z_3=-1,\ w_1=i,\ w_2=0,\ w_3=-i$

Since bilinear transformation preserves cross-ratio, we have

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)} = \frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_5)}$$

$$\frac{(1-i)(-1-z)}{(i+1)(z-1)} = \frac{(i-0)(-i-w)}{(0+i)(w-i)}$$

$\Rightarrow$ $w=\frac{1+iz}{1-iz}$ which is the required B.T.

Invariant pts are solution of $w=z$

$$z=-\frac{1}{2}\left(1+i\pm\sqrt{6i}\right)$$

(5) Given $f(z)=\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1}+\frac{1}{z-2}$

$$= \frac{1}{z-2}-\frac{1}{z-1}$$

(i) $1<|z|<2$

$$f(z) = \frac{1}{z-2}-\frac{1}{z-1}$$

$$= -\frac{1}{2}\,\frac{1}{1-\frac{z}{2}} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= -\frac{1}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right) - \frac{1}{z}\left(1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\cdots\right) \text{ for } |z|<2,\ |z|>1$$

$$= -2\left(2+\frac{z}{2}+\frac{z^2}{2^2}+\cdots\right) - \left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) \text{ for } |z|<2,\ |z|>1$$

$$= -2\left(2+z+\frac{z^2}{2^2}+\cdots\right) - \left(z^{-1}+z^{-2}+z^{-3}+\cdots\right) \text{ for } 1<|z|<2$$

(ii) $0<|z-1|<1$

$$f(z) = \frac{1}{z-1-1}-\frac{1}{z-1} = -\left(1+(z-1)+(z-1)^2+\cdots\right)-(z-1)^{-1}$$

for $0<|z-1|<1$