

Continous Assessment Test - II Solution
 Fall 2025-26
 BAPHY105 (SCORE)
 Slot: C1+TC1

Q.1 Discuss the sequential Stern-Gerlach experiment with appropriate diagram and explain how it demonstrates the non-commutativity of spin measurements along different axes and the probabilistic nature of quantum measurement.

Solution:

- **Setup (6 marks):** A collimated silver atom beam passes through an inhomogeneous magnetic field along z -axis (SG1), splitting into eigenstates

$$|+z\rangle, \quad |-z\rangle$$

Passing $|+z\rangle$ through SG2 along x -axis gives

$$|+x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle + |-z\rangle), \quad |-x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle - |-z\rangle)$$

Again passing $|+x\rangle$ through SG3 along z -axis gives

$$|+z\rangle = \frac{1}{\sqrt{2}}(|+x\rangle + |-x\rangle), \quad |-z\rangle = \frac{1}{\sqrt{2}}(|+x\rangle - |-x\rangle)$$

Draw a neat diagram showing initial z -axis splitting, x -axis splitting, and final z -axis splitting, labeling all eigenstates.

- **Probabilistic nature (2 marks):** $|+z\rangle$ through x -axis SG2 splits into $|+x\rangle$ and $|-x\rangle$ (50% each). Passing these through z -axis SG3 again splits into $|+z\rangle$ and $|-z\rangle$ (50% each).
- **Non-commutativity (2 mark):** measuring spin along z -axis (S_z) and then along x -axis (S_x) gives results that depend on the order of measurements. The second measurement disturbs the first, showing that $[S_z, S_x] \neq 0$. Hence, spin components along different axes cannot be simultaneously measured with definite values.

Q.2 Consider a normalized wave function

$$\psi(x) = \sqrt{\frac{30}{L^5}} x(L-x), \quad 0 < x < L$$

Calculate the expectation value $\langle x \rangle$, $\langle x^2 \rangle$, and the uncertainty Δx .

Solution: Given wavefunction

$$\psi(x) = \sqrt{\frac{30}{L^5}} x(L-x), \quad 0 < x < L$$

Expectation value $\langle x \rangle$ is given by,

$$\langle x \rangle = \int_0^L x |\psi(x)|^2 dx = \frac{30}{L^5} \int_0^L x \cdot x^2 (L-x)^2 dx = \frac{30}{L^5} \int_0^L x^3 (L-x)^2 dx$$

Now,

$$\int_0^L x^3(L-x)^2 dx = \int_0^L x^3(L^2 - 2Lx + x^2) dx = \int_0^L (L^2x^3 - 2Lx^4 + x^5) dx$$

Integrating each term one by one we get,

$$\int_0^L L^2x^3 dx = \frac{L^2L^4}{4} = \frac{L^6}{4}, \quad \int_0^L 2Lx^4 dx = 2L \cdot \frac{L^5}{5} = \frac{2L^6}{5}, \quad \int_0^L x^5 dx = \frac{L^6}{6}$$

So the total integral become

$$\int_0^L x^3(L-x)^2 dx = \frac{L^6}{4} - \frac{2L^6}{5} + \frac{L^6}{6} = \frac{L^6}{60}$$

So

$$\langle x \rangle = \frac{30}{L^5} \cdot \frac{L^6}{60} = \frac{L}{2} \quad (4 \text{ marks})$$

Expectation value $\langle x^2 \rangle$ is given by,

$$\langle x^2 \rangle = \int_0^L x^2 |\psi(x)|^2 dx = \frac{30}{L^5} \int_0^L x^2 \cdot x^2(L-x)^2 dx = \frac{30}{L^5} \int_0^L x^4(L-x)^2 dx$$

Computing the integral

$$\int_0^L x^4(L-x)^2 dx = \int_0^L (L^2x^4 - 2Lx^5 + x^6) dx = L^2 \frac{L^5}{5} - 2L \frac{L^6}{6} + \frac{L^7}{7} = \frac{L^7}{5} - \frac{L^7}{3} + \frac{L^7}{7} = \frac{L^7}{105}$$

Hence,

$$\langle x^2 \rangle = \frac{30}{L^5} \cdot \frac{L^7}{105} = \frac{30L^2}{105} = \frac{2}{7}L^2 \quad (4 \text{ marks})$$

Uncertainty Δx is

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{2}{7}L^2 - \frac{L^2}{4}} = \frac{L}{2\sqrt{7}} \quad (2 \text{ marks})$$

Q.3 (a) Explain the physical significance of non-commuting operators in quantum mechanics. Derive the commutation relation between the position operator \hat{x} and momentum operator \hat{p}_x

Solution: Physical significance (1 marks): Non-commuting operators represent physical quantities that cannot be measured simultaneously with exact precision (like position and momentum). Their non-commutation leads to the uncertainty principle.

Derivation of commutation relation between position and momentum (4 marks)

We want to compute the commutator

$$[\hat{x}, \hat{p}_x] = \hat{x}\hat{p}_x - \hat{p}_x\hat{x}$$

Let $\psi(x)$ be an arbitrary wavefunction. Then

$$\hat{x}\hat{p}_x\psi(x) = x \cdot \left(-i\hbar \frac{d}{dx} \psi(x) \right) = -i\hbar x \frac{d\psi(x)}{dx}$$

On the other hand

$$\hat{p}_x\hat{x}\psi(x) = -i\hbar \frac{d}{dx} (x\psi(x)) = -i\hbar \left(\psi(x) + x \frac{d\psi(x)}{dx} \right)$$

Therefore, the action of the commutator on $\psi(x)$ is

$$[\hat{x}, \hat{p}_x]\psi(x) = \hat{x}\hat{p}_x\psi(x) - \hat{p}_x\hat{x}\psi(x) = -i\hbar x \frac{d\psi(x)}{dx} - \left[-i\hbar \left(\psi(x) + x \frac{d\psi(x)}{dx} \right) \right]$$

Simplifying

$$[\hat{x}, \hat{p}_x]\psi(x) = -i\hbar x \frac{d\psi(x)}{dx} + i\hbar\psi(x) + i\hbar x \frac{d\psi(x)}{dx}$$

The terms $-i\hbar x \frac{d\psi(x)}{dx}$ and $+i\hbar x \frac{d\psi(x)}{dx}$ cancel, leaving

$$[\hat{x}, \hat{p}_x]\psi(x) = i\hbar\psi(x)$$

Since this holds for any $\psi(x)$, the operator relation is

$$[\hat{x}, \hat{p}_x] = i\hbar.$$

Q.3 (b) Solution:

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & -2 & 3 \end{pmatrix}.$$

Eigenvalues (2 marks) Since M is lower triangular, its eigenvalues are the diagonal entries

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3.$$

Eigenvectors (3 marks) ,

Solve $(M - \lambda I)\mathbf{v} = 0$ for each eigenvalue.

For $\lambda_1 = 1$

$$M - I = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 2 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

This gives the system of linear equations

$$\begin{cases} x + y = 0, \\ 2x - 2y + 2z = 0. \end{cases}$$

From the first equation, $y = -x$. Substitute into the second eq

$$2x - 2(-x) + 2z = 0 \Rightarrow 4x + 2z = 0 \Rightarrow z = -2x.$$

Hence,

$$\mathbf{v}_1 = x \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \quad \text{for } x = 1.$$

Normalizing the vector

$$\|\mathbf{v}_1\| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6},$$

$$\hat{\mathbf{v}}_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}.$$

For $\lambda_2 = 2$

$$M - 2I = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & -2 & 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

System of equations

$$\begin{cases} -x = 0, \\ x = 0, \\ 2x - 2y + z = 0. \end{cases}$$

From the first two, $x = 0$. Substitute into the third:

$$-2y + z = 0 \Rightarrow z = 2y.$$

Thus,

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

Normalizing

$$\|\mathbf{v}_2\| = \sqrt{0^2 + 1^2 + 2^2} = \sqrt{5},$$

$$\hat{\mathbf{v}}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

For $\lambda_3 = 3$

$$M - 3I = \begin{pmatrix} -2 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -2 & 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

System of equations

$$\begin{cases} -2x = 0, \\ x - y = 0, \\ 2x - 2y = 0. \end{cases}$$

From the first, $x = 0$; from the second, $y = 0$. The third is automatically satisfied, and z is free (take $z = 1$).

Hence,

$$\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Q. 4 (a) An electron in a 1D infinite potential well of length $L = 5\text{\AA}$ makes a transition from the 6th to the 4th excited state. Calculate the frequency and wavelength of the emitted photon.

Solution (5 marks)

In a 1D infinite square well the energy levels are

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}, \quad n = 1, 2, 3, \dots$$

The " k -th excited state" corresponds to quantum number $n = k + 1$. So the 6th excited state has $n_1 = 6 + 1 = 7$ and the 4th excited state has $n_2 = 4 + 1 = 5$.

Energy of the emitted photon is the difference

$$\Delta E = E_7 - E_5 = \frac{\pi^2 \hbar^2}{2mL^2} (7^2 - 5^2) = \frac{12\pi^2 \hbar^2}{mL^2} = \frac{3h^2}{mL^2}.$$

The frequency ν and wavelength λ of the emitted photon are

$$\nu = \frac{\Delta E}{h} = \frac{3h}{mL^2}, \quad \lambda = \frac{c}{\nu},$$

where h is Planck's constant, m the electron mass, and c the speed of light.

Substitute numerical values,

$$h = 6.626 \times 10^{-34} \text{ Js}, \quad m = 9.1 \times 10^{-31} \text{ kg}, \quad L = 5.0 \times 10^{-10} \text{ m}, \quad c = 3 \times 10^8 \text{ m/s}.$$

$$\nu = \frac{3h}{mL^2} \approx \frac{3(6.626 \times 10^{-34})}{(9.1 \times 10^{-31})(5.0 \times 10^{-10})^2} \approx 8.73 \times 10^{15} \text{ Hz}.$$

Wavelength,

$$\lambda = \frac{c}{\nu} \approx \frac{3 \times 10^8}{8.73 \times 10^{15}} \approx 3.43 \times 10^{-8} \text{ m}.$$

Q. 4(b) In a cubic infinite potential well of side $L = 2\text{nm}$, an electron occupies the quantum state $\psi_{2,1,1}(x, y, z)$. Calculate the quantized wavelength $\lambda_x, \lambda_y, \lambda_z$ along each axis and the total energy E in eV.

Solution (5 marks)

Given cubic infinite well with side $L = 2.0 \text{ nm} = 2.0 \times 10^{-9} \text{ m}$, $\psi_{2,1,1}(x, y, z) (\Rightarrow n_x = 2, n_y = 1, n_z = 1)$.

Quantized wavelengths along each axis: For a particle in a 1D infinite well of length L the standing wave wavelength corresponding to quantum number n is obtained from the quantization condition,

$$k = \frac{n\pi}{L}$$

which gives,

$$\lambda = \frac{2L}{n}.$$

Thus

$$\begin{aligned}\lambda_x &= \frac{2L}{n_x} = \frac{2L}{2} = L = 2.0 \text{ nm}, \\ \lambda_y &= \frac{2L}{n_y} = \frac{2L}{1} = 2L = 4.0 \text{ nm}, \\ \lambda_z &= \frac{2L}{n_z} = \frac{2L}{1} = 2L = 4.0 \text{ nm}.\end{aligned}$$

Total energy: The energy levels for a particle in a 3D cubic infinite well are

$$E_{n_x, n_y, n_z} = \frac{h^2}{8mL^2}(n_x^2 + n_y^2 + n_z^2),$$

Substitute $n_x = 2$, $n_y = 1$, $n_z = 1$:

$$n_x^2 + n_y^2 + n_z^2 = 2^2 + 1^2 + 1^2 = 4 + 1 + 1 = 6.$$

So

$$E = \frac{h^2}{8mL^2} \times 6.$$

Using

$$h = 6.626 \times 10^{-34} \text{ Js}, \quad m = 9.1 \times 10^{-31} \text{ kg}, \quad L = 2.0 \times 10^{-9} \text{ m},$$

we get

$$E \approx 9.04 \times 10^{-20} \text{ J}.$$

Convert to electronvolts ($1 \text{ eV} \approx 1.6 \times 10^{-19} \text{ J}$)

$$E \approx \frac{9.04 \times 10^{-20}}{1.6 \times 10^{-19}} \approx 0.565 \text{ eV}.$$

Q.5 Derive the expression for the energy eigenvalue E_n of an electron confined in a 1D box of length L . In an experiment, the values of $\sqrt{E_n}$ (with E_n in eV) are plotted against quantum number n , resulting in a straight line of slope $A = 0.1 \text{ eV}^{1/2}$. Determine the length L of the box in nanometer.

Solution:

Full derivation of the energy eigenvalues, starting from the time-independent Schrodinger equation, including its general solution, application of boundary conditions, and the relevant diagram, carries **6 marks**.

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} = \frac{n^2h^2}{8mL^2}$$

As $\sqrt{E_n} \propto n$, so the plot of $\sqrt{E_n}$ vs n is a straight line starting from $n = 1$, with slope

$$A = \frac{h}{\sqrt{8mL}}$$

Solving for Length (L), we get

$$L = \frac{h}{\sqrt{8mA}}.$$

Given, $A = 0.1$ in unit of $\text{eV}^{1/2}$. Converting to SI unit (using $1 \text{ eV} \approx 1.6 \times 10^{-19} \text{ J}$)

$$A \approx 0.1 \times (1.6 \times 10^{-19} \text{ J})^{1/2} = 0.4 \times 10^{-10} \text{ J}^{1/2}.$$

Thus,

$$L = \frac{6.626 \times 10^{-34}}{\sqrt{8 \times 9.1 \times 10^{-31} \times 0.4 \times 10^{-10}}} \approx 0.614 \times 10^{-8} \text{ m}$$

$$L \approx 6.14 \text{ nm}. \quad (4 \text{ marks})$$