Multivariable Calculus and Differential Equations CAT-1 D2 SLOT KEY

Question 1

If $u = \frac{x+y}{x-y}$, $v = \frac{xy}{(x-y)^2}$ verify whether u, v are functionally dependent. If so, find the relation between them.

Solution:

$$\frac{\partial u}{\partial x} = \frac{(x-y)\cdot 1 - (x+y)\cdot 1}{(x-y)^2} = \frac{-2y}{(x-y)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x-y)\cdot 1 - (x+y)(-1)}{(x-y)^2} = \frac{2x}{(x-y)^2}$$

$$\frac{\partial v}{\partial x} = y\cdot\frac{(x-y)^2\cdot 1 - x\cdot 2(x-y)}{(x-y)^4} = -\frac{y(x+y)}{(x-y)^3}$$

$$\frac{\partial v}{\partial y} = x\cdot\frac{(x-y)^2\cdot 1 - y\cdot 2(x-y)(-1)}{(x-y)^4} = \frac{x(x+y)}{(x-y)^3}$$

Thus, $\longrightarrow$ 4M

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{-2xy(x+y)}{(x-y)^5} + \frac{2xy(x+y)}{(x-y)^5} = 0.$$

Hence, u, v are functionally dependent. $\longrightarrow$ 3M

To find the relation we use the following algebraic identity:

$$(x+y)^2 - (x-y)^2 = 4xy.$$

$$\frac{(x+y)^2}{(x-y)^2} - 1 = \frac{4xy}{(x-y)^2}$$

$$u^2 - 1 = 4v.$$

$\longrightarrow$ 3M

Question 2

Expand $\log(1+x+y)$ about (0,0) up to the third term using Taylor's series.

Solution:

Taylor series expansion of $f(x,y)$ up to third degree term is

$$f(x,y) = f(0,0) + \frac{1}{1!}\left[xf_x(0,0) + yf_y(0,0)\right]$$
$$+\frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$
$$+\frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right].$$

Let $f(x,y) = \log(1+x+y)$. Then $f(0,0) = \log 1 = 0$.

$\longrightarrow$ $\boxed{3M}$

We compute the derivatives:

$$f_x(x,y) = \frac{1}{1+x+y} \quad f_x(0,0) = 1,$$

$$f_y(x,y) = \frac{1}{1+x+y} \quad f_y(0,0) = 1,$$

$$f_{xx}(x,y) = -\frac{1}{(1+x+y)^2} \quad f_{xx}(0,0) = -1,$$

$$f_{xy}(x,y) = -\frac{1}{(1+x+y)^2} \quad f_{xy}(0,0) = -1,$$

$$f_{yy}(x,y) = -\frac{1}{(1+x+y)^2} \quad f_{yy}(0,0) = -1,$$

$$f_{xxx}(x,y) = \frac{2}{(1+x+y)^3} \quad f_{xxx}(0,0) = 2,$$

$$f_{xxy}(x,y) = \frac{2}{(1+x+y)^3} \quad f_{xxy}(0,0) = 2,$$

$$f_{xyy}(x,y) = \frac{2}{(1+x+y)^3} \quad f_{xyy}(0,0) = 2,$$

$$f_{yyy}(x,y) = \frac{2}{(1+x+y)^3} \quad f_{yyy}(0,0) = 2.$$

$\boxed{\text{5M}}$

Taylor series expansion of $f(x, y) = log(1 + x + y)$ upto third degree term is given by

$$\log(1 + x + y) = (x + y) - \frac{1}{2}(x^2 + 2xy + y^2) + \frac{1}{3}(x^3 + 3x^2y + 3xy^2 + y^3).$$

$$log(1 + x + y) = \frac{(x + y)}{1} - \frac{(x + y)^2}{2} + \frac{(x + y)^3}{3} - \cdots$$

$\boxed{\text{2M}}$

Question 3

A thin closed rectangular box is to have one edge equal to twice the other, and a constant volume $72\, m^3$. Find the least surface area of the box.

Solution:

Let the sides of the rectangular box be $2x,\ x,\ y$.

Then the volume is

$$2x \cdot x \cdot y = 72$$

$$2x^2y = 72, \quad \Longrightarrow \quad x^2y = 36.$$

The surface area is given by

$$S = 2(2x \cdot x) + 2(2x \cdot y) + 2(x \cdot y) = 4x^2 + 6xy.$$

Now we find the minimum of surface area $S = 4x^2 + 6xy$ under the condition $x^2y = 36$.

$\boxed{\text{3M}}$

Let

$$f = 4x^2 + 6xy - -(1), \quad g = x^2y - 36 - -(2).$$

Let the auxiliary function 'F' be $F = f + \lambda g$

$$F(x, y) = (4x^2 + 6xy) + \lambda(x^2y - 36). - - (3)$$

By Lagrange's method the values of x, y, z for which 'F'is minimum is obtained from the following equations:

$$\frac{\partial F}{\partial x} = 0 \quad \Longrightarrow \quad 8x + 6y + 2\lambda xy = 0, - - (4)$$

$$\frac{\partial F}{\partial y} = 0 \quad \Longrightarrow \quad 6x + \lambda x^2 = 0, -- (5)$$

$$\frac{\partial F}{\partial z} = 0 \quad \Longrightarrow \quad 0 = 0,$$

$$\frac{\partial F}{\partial \lambda} = 0 \quad \Longrightarrow \quad x^2 y - 36 = 0. -- (6)$$

$\longrightarrow \boxed{3M}$

From (5):

$$x = \frac{-6}{\lambda}. -- (7)$$

Substituting in (6):

$$y = \frac{36}{x^2} = \lambda^2. -- (8)$$

$\longrightarrow \boxed{2M}$

Now substitute $x = -6/\lambda,\ y = \lambda^2$ into (4):

$$\frac{-48}{\lambda} + 6\lambda^2 - 12\lambda^2 = 0 \quad \Longrightarrow \quad \lambda^3 = -8.$$

Thus,

$$\lambda = -2.$$

Substituting $\lambda = -2$ in (7) and (8), we get

$$x = 3, \quad y = 4.$$

Therefore, S has a minimum value at $(x, y) = (3,4)$.

The minimum surface area is

$$S = 4(3)^2 + 6(3)(4) = 108.$$

$\longrightarrow \boxed{2M}$

Question 4

Evaluate $\iint\limits_D (x - y)\, dx\, dy$ over a region bounded by curves $x = y^2$ and $x = \frac{y^2}{2} + 1$. Sketch a diagram of the region.

Solution: From the system of equations of $x = y^2$ and $x = y^2/2 + 1$ we obtain two intersection points: $(-\sqrt{2}, 2)$ and $(\sqrt{2}, 2)$. Region D is horizontally simple, so:

$$\iint_D (x - y)\, dx\, dy = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{y^2}^{y^2/2+1} (x - y)\, dx\, dy$$

$$\longrightarrow \boxed{\text{3M}}$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \left[\frac{x^2}{2} - xy\right]_{x=y^2}^{x=y^2/2+1} dy$$

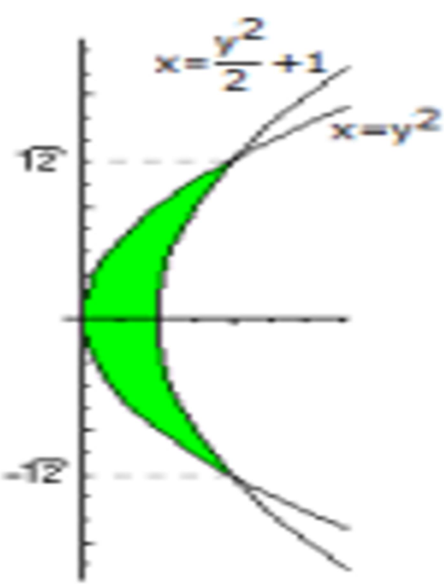
$$= \int_{-\sqrt{2}}^{\sqrt{2}} \left(\frac{(y^2/2+1)^2}{2} - y(y^2/2+1) - \frac{y^4}{2} + y^3\right) dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} (3y^4/8 + y^2/2 + 1/2 - y)\, dy$$

$$= [3y^5/40 + y^3/6 + y/2 - y^2/2]_{-\sqrt{2}}^{\sqrt{2}} = \frac{16\sqrt{2}}{15}.$$

$$\longrightarrow \boxed{\text{5M}}$$

Sketch of the region:



$$\longrightarrow \boxed{\text{2M}}$$

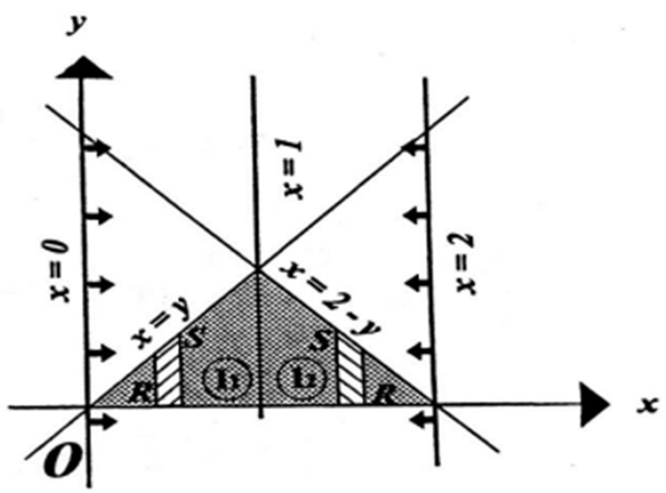
Question 5

Change the order of integration in $\int_0^1 \int_y^{2-y} x\, y dx dy$ and hence evaluate;

Solution:

It is correct form   X: y → 2-y; Y:0 → 1

The region is bounded by x=y, x+y=2,y=0, y=1



2M

Now divide the region into two parts i.e. R_1 and R_2. Changed order is $dy\, dx$. Draw horizontal strip.

For region R_1:

Limits are $x: 0 \to 1$,
$y: 0 \to x$.

$$\int_0^1 \int_y^{2-y} x\, y\, dx\, dy = \int_0^1 \int_0^x x\, y\, dy\, dx$$

$$\int_0^1 \left[\frac{xy^2}{2}\right]_0^x dx$$

$$= \int_0^1 \left[\frac{x^3}{3-0}\right] dx$$

$$\frac{1}{2}\int_0^1 x^3\, dx = \frac{1}{2}\left[\frac{x^4}{4}\right]_0^1$$

$$\frac{1}{8}[x^4]_0^1 = \frac{1}{8}[1-0]$$

$$= \frac{1}{8}$$

$\longrightarrow$ $\boxed{\text{3M}}$

For region R_2: Limits are $x: 1 \to 2$,
$y: 0 \to 2-x$.

$$\int_1^2 \int_0^{2-x} x\, y\, dy\, dx = \int_1^2 \left[\frac{xy^2}{2}\right]_0^{2-x} dx = \frac{1}{2}\int_1^2 x\,(2-x)^2\, dx$$

$$= \frac{1}{2}\int_1^2 (\,4x - 4x^2 + x^3)\, dx$$

$$= \frac{1}{2}\left[2x^2 - \frac{4x^3}{3} + \frac{x^4}{4}\right]_1^2$$

$$= \frac{1}{2}\big((8 - 32/3 + 4) - (2 - 4/3 + 1/4)\big)$$

$$= \frac{1}{2}\left(8 + 4 - \frac{32}{3} - 2 - \frac{1}{4} + \frac{4}{3}\right)$$

$$\frac{1}{2}[\frac{5}{3}] = \frac{5}{24}$$

$\longrightarrow$ $\boxed{\text{3M}}$

$$R = R_1 + R_2$$

$$\frac{1}{8} + \frac{5}{24}$$

$$= \frac{1}{3}.$$

$\longrightarrow$ $\boxed{\text{2M}}$